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On The Hypersurface of a Finsler Space with Some Special Cases

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Abstract

The study of special Finsler spaces has been introduced by M.Matsumoto [4]. The purpose of the present paper is to study hypersurfaces of special Finsler spaces and also to investigate the various kinds of hypersurfaces of Finsler space with special (α,β) metric.

Keywords: *Hypersurfaces, finsler spaces, C-reducible, quasi-C-reducible, preducible.*

1. INTRODUCTION

The study of spaces endowed with generalized metrics was initiated by P. Finsler in1918.The theory of hypersurfaces in general depends to a large extent on the study of the behavior of curves in them. The authors G.M. Brown, Moor, C. Shibata, M. Matsumoto, B.Y. Chen, C.S. Bagewadi, L.M. Abatangelo, Dragomir and S. Hojo have studied different properties of subspaces of Finsler, Kahler and Riemannian spaces. The concept of the (α,β)-metric $L(\alpha,\beta)$ was introduced by M. Matsumoto [5] and has been studied by many authors [1], [2], [12]. The study of some well known (α,β)-metrics, the Randers metric $\alpha+\beta$,the Kropina metric α^2/β and the generalized Kropina metric $\alpha^{(m+1)}/\beta^m$ have greatly contributed to the growth of Finsler geometry and its applications to theory of relativity.

2. PRELIMINARY NOTES

Let $F^n = (M^n,L)$ be a Finsler space on a differentiable manifold M endowed with a fundamental function $L(x,y)$. By a Finsler space, we mean a triple$F^n = (M,D,L)$, where M denotes n-dimensional differentiable manifold,D is an open subset of a tangent vector bundle TM endowed with the differentiable structure induced by the differentiable structure of the manifold TM and L : $D \to R$ is a differentiable mapping having the properties

1. $L(x,y) > 0$, for $(x,y) \in D$,
2. $L(x,\lambda) = |\lambda| L(x,y)$,for any $(x,y) \in D$and $\lambda \in R$ such that $(x,\lambda y) \in D$,
3. The d-tensor field$g_{ij} = \frac{1}{2}(\partial_i\partial_j L^2)$, for $(x,y) \in D$, where $\partial_i = \frac{\partial}{\partial y^i}$, is non degenerate on D.

We have the following identities (see [1] and [11]):

(a) $g_{ij} = \frac{1}{2}\partial_i\partial_j L^2, g^{ij} = (g_{ij})^{(-1)}, \partial_i = \partial/(\partial y^i)$

(b) $C_{ijk} = \frac{1}{2}\partial_k g_{ij}, C^k_{ij} = \frac{1}{2}g^k m(\partial_m g_{ij}), h_{ij} = g_{ij} - l_i l_j$

(c) $\gamma^i_{jk} = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk}$ (1)

(d) $G^i = \frac{1}{2}\gamma^i_{jk}y^j y^k, G^i_j = \partial_j G^i$

(e) $G^i_{jk} = \partial_k G_(j,)G^i_{jkl} = \partial_i G^i_{jk}$

(f) $F^i_{jk} = \frac{1}{2}g^{ir}(\delta_j g_{rk} + \delta_k g_{rj} - \delta_r g_{jk})$

(g) $Phijk = u(hi)Cijk/h + ChjrCik/r\,0$,

(h) $Shijk = u(jk)\{ChkrCijr\}$,

Where $\delta_j = \partial_j - G^r_j\partial_r$, the index 0 means contraction by y^i and the notation $u_{(jk)}$ denotes the interchange of indices j,k and substraction. Also, we are devoted to a special Finsler space $F^n = (M^n,L)$ with a metric

$$L(\alpha\beta) = \alpha + \beta + \frac{\beta^{(n+1)}}{\alpha^n} \quad (2)$$

$$L_\alpha = \frac{\alpha^{(n+1)} - n\beta^{(n+1)}}{\alpha^{(n+1}}), L_\beta = \frac{(\alpha^n + (n+1)\beta^n)}{\alpha^n}$$

$$L_{\alpha\alpha} = \frac{n(n+1)\beta^{(n+1)}}{\alpha^{(n+2)}}, L_{\beta\beta} = \frac{n(n+1)\beta^{(n-1)}}{\alpha^{n}}$$

$$L_{\alpha\beta} = \frac{-n(n+1)\beta^{n}}{\alpha^{n+1}},$$

Where

$$L_{\alpha} = \frac{\partial L}{\partial \alpha}, L_{\beta} = \frac{\partial L}{\partial \beta}, L_{\alpha\alpha} = \frac{\partial L_{\alpha}}{\partial \alpha}$$

$$L_{\beta\beta} = \frac{\partial L_{\beta}}{\partial \beta} \; and \; L_{\alpha\beta} = \frac{\partial L_{\alpha}}{\partial \beta}$$

In the special Finsler space $F^n = (M^n, L)$ the normalized element of support $l_i = \partial_i L$ And the angular metric tensor h_{ij} are given by [10]:

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i,$$

$$Y_i = a_{ij} y^j,$$

$$p = L L_{\alpha\alpha^{-1}} = \frac{(\alpha^{n+1} + \alpha\beta^{n} + \beta^{n+1})(\alpha^{n+1} - \beta^{n+1})}{\alpha^{2(n+1)}}$$

$$q_0 = L L_{\beta\beta} = \frac{n(n+1)\beta^{n-1}(\alpha^{n+1} + \alpha\beta^{n} + \beta^{n+1})}{\alpha^{2n}},$$

$$q_1 = L L_{\alpha\beta}\, \alpha^{-1} = \frac{-n(n+1)\beta^{n}(\alpha^{n+1} + \alpha\beta^{n} + \beta^{n+1})}{\alpha^{(2n+1)}}$$

$$q_2 = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha \alpha^{-1}),$$

$$= \left(\frac{\alpha^{n+1} + \alpha\beta^{n} + \beta^{n+1}}{\alpha^{n+2}}\right)\left(\frac{n(n+2)\beta^{n+1} - \alpha^{n+1}}{\alpha^{n+2}}\right)$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_1(b_i Y_j + b_j Y_i) + q_2 Y_i Y_j \quad (3)$$ Where (4)

The fundamental tensor $g_{ij} = \frac{1}{2}\partial_i \partial_j L^2$ and its reciprocal tensor g^{ij} is given by [10]

$$g_{ij} = p a_{ij} + p_0 b_i b_j + p_1(b_i Y_j + b_j Y_i) + p_2 Y_i Y_j, \quad (5)$$ Where

$$p_0 = q_0 + L_\beta^2 = \frac{n(n+1)\beta^{n-1}(\alpha^{n+1} + \alpha\beta^{n} + \beta^{n+1}) + (\alpha^{n} + (n+1)\beta^{n})^2)}{\alpha^{2n}}$$

$$p_1 = q_1 + L^{-1} p L_\beta ,$$

$$= \frac{-n(n+1)\beta^{n}(\alpha^{n+1} + \alpha\beta^{n} + \beta^{n+1}) + (\alpha^{n+1} - \beta^{n+1})(\alpha^{n} + (n+1)\beta^{n}}{\alpha^{2(n+1)}}, \quad (6)$$

$$p_2 = q_2 + p^2 L^{-2},$$

$$= \frac{(\alpha^{n+1} + \alpha\beta^{n} + \beta^{n+1})(n(n+2)\beta^{n+1} - \alpha^{n+1}) + (\alpha^{n+1} - \beta^{n+1})^2}{\alpha^{2(n+2)}}$$

$$g_{ij} = p^{-1} a^{ij} - S_0 b^i b^j - S_1(b^i y^j + b^j y^i) - S_2 y^i y^j, \quad (7)$$

Where

$$b^i = a^{ij} b_j, S_0 = (p p_0 + (p_0 p_2 - p_1^2)\ \alpha^2)/\zeta p$$

$S_1 = (p p_1 + (p_0 p_2 - p_1^2)\beta)/\zeta p,$ (8)

$S_2 = (p p_2 + (p_0 p_2 - p_1^2) b^2)/\zeta p, b^2 = a_{ij} b^i b^j,$

$\zeta = p(p + p_0 b^2 + p_1) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2)$

3. HYPERSURFACE OF THE SPECIAL FINSLER SPACES

Now we consider the special Finsler spaces like P-reducible, quasi-C-reducible, and C-reducible. Then we prove all these special Finsler space are well-defined in Finsler hypersurface F^{n-1} under some conditions.

Definition 3.1 (see [7]) A Finsler space F^n is called a P-reducible, if the torsion tensor P_{ijk} is written as

$$P_{ijk} = (h_{ij}P_k + h_{jk}P_i + h_{ki}P_j)/(n + 1) \quad (9)$$

Where

$$P_i = P^m_{im} = C_{i/0}$$

Contracting (8) by $B^{ijk}_{\alpha\beta\gamma}$ and using $h_{\alpha\beta} = g_{\alpha\beta} - l_\alpha l_\beta$, and $h_{\alpha\beta} = h_{ij} B^{ij}_{\alpha\beta}$ we obtain

$$P_{ijk} B^{ijk}_{\alpha\beta\gamma} = (h_{ij}P_k + h_{jk}P_i + h_{ki}P_j) B^{ijk}_{\alpha\beta\gamma} /(n + 1),$$

$$P_{ijk} B^{ijk}_{\alpha\beta\gamma} = (h_{\alpha\beta}P_\gamma + h_{\beta\gamma}P_\alpha + h_{\gamma\alpha}P_\beta)/n,$$

where we set. $P_i B^i_\alpha = P_\alpha = C_{\alpha/0}$ Hence we have the following result.

Theorem 3.2. A hypersurface of a P-reducible Finsler space is P-reducible.

Next we consider the curvature tensor of F^n

Shijk = ChkrCijr −ChjrCikr .

Contracting above equation by $B_{\delta\alpha\beta\gamma}^{hijk}$ and using (1), we have

$$S_{hijk}B^{hijk}_{\delta\alpha\beta\gamma} = (C_{hkr}C^r_{ij} - C_{hjr}C^r_{ik})B^{hijk}_{\delta\alpha\beta\gamma}$$

= *ChkrCijr Bδαβγhijk −ChjrCikr Bδαβγhijk ,*

Sδαβγ = CδγθCαβθ −CδβθCαγθ.

Hence $S_{\delta\alpha\beta\gamma}$ is the curvature tensor of F^{n-1}.

Definition 3.3 (see [9]) A Finsler space F^n (n >2) is called a quasi-Creducible, if the torsion tensor C_{ijk} is written as

Cijk = AijCk + AjkCi + AkiCj, (10)

where A_{ij} is a symmetric Finsler tensor field satisfying $A_{io} = A_{ij}y^j = 0$. Contracting (9) by projection factor $B_{\alpha\beta\gamma}^{ijk}$, we obtain

CijkBαβγijk = (AijCk + AjkCi + AkiCj)Bαβγijk ,

$$C_{ijk}B^{ijk}_{\alpha\beta\gamma} = A_{ij}B^{ij}_{\alpha\beta}C_kB^k_\gamma + A_{jk}B^{jk}_{\beta\gamma}C_iB^i_\alpha + A_{ki}B$$

.

By using the notations on Finsler hypersurface (see[3],[8],[11]): We obtain

Cαβγ = AαβCγ + AβγCα + AγαCβ,

where we setting $A_{\alpha\beta} = A_{ij}B_{\alpha\beta}^{ij}$ is a symmetric Finsler tensor field on hypersurface F^{n-1}. Thus we have:

Theorem 3.4 A hypersurface F^{n-1} of a quasi-C-reducible Finsler space F^n is quasi-C-reducible.

Suppose we assume that $C_\alpha = 0$, that implies

$$C_iB^i_\alpha = 0 \quad (11)$$

it means that C_i is tangential to the hypersurface F^{n-1}. then from (9), we have $C_{\alpha\beta\gamma} = 0$ therefore by Deickes theorem the quasi-C-reducible Finsler hypersurface is Riemannian, which proves the following:

Theorem 3.5 A quasi-C-reducible Finsler hypersurface F^{n-1} is Riemannian, if the vector C_i is tangential to hypersurface F^{n-1}.

Definition 3.6 (see [6]) A Finsler space F^n(n >2) is said to be C-reducible, if it satisfies the equation

(n + 1)Cijk = hijCk + hjkCi + hkiCj, (12)

where *Ci = gjkCijk*

Contracting (11) by $B_{\alpha\beta\gamma}^{ijk}$ and using using the notations on Finsler hyper surface (see [3], [8],[11]): we obtain

nCαβγ = hαβCγ + hβγCα + hαγCβ, (13)

Where. $C_\alpha = C_iB^i_\alpha = g^{\beta\gamma}C_{\alpha\beta\gamma}$ Hence we have:

Theorem 3.7 *(see [13]) A hypersurface of a C-reducible Finsler space is C-reducible.*

Using the condition (10) in (12), we state that the following result:

Theorem 3.8 A hypersurface F^{n-1} of a C-reducible Finsler space is Riemannian, if the torsion vector C_i is tangential to hypersurface F^{n-1}.

4. HYPERSURFACE F^{n-1} (C) OF THE SPECIAL FINSLER SPACE

Let us consider special Finsler metric $L = \alpha + \beta + \frac{\beta^{n+1}}{\alpha^n}$

with a gradient $b_i(x) = \partial_i b$ for a scalar function b(x) and a hypersurface F^{n-1} (c)given by the equation b(x) = c (constant) [14].

From parametric equation $x^i = x^i(u^\alpha)$ of F^{n-1} (c), we get $\partial_\alpha b(x(u)) = 0 =$

$b_iB^i_\alpha$, so that$b_i(x)$ are regarded as covariant components of a normal vector field of$F^{n-1}(c)$. Therefore, along the F^{n-1} (c) we have

$$b_iB^i_\alpha = 0 \; and \; b_iy^i = 0 \quad (14)$$

The induced metric L(u,v) of F^{n-1} (c) is given by

$$L(u,v) = a_{\alpha\beta}v^\alpha v^\beta \;, a_{\alpha\beta} = a_{ij}B^i_\alpha B^j_\beta \quad (15)$$

which is a Riemannian metric.

At a point of F^{n-1} (c), from (4),(6) and (8), we have

p = 1,q0 = 0,q1 = 0,q2 = −α−2,p0 = 1,p1 = α−1, p2 = 0,ζ = 1,S0 =

0,$S_1 = \alpha^{-1}$,$S_2 = -b^2/\alpha^2$. (16)

Therefore from (7) we get

$$g^{ij} = a^{ij} - \frac{1}{\alpha}(b^iy^j + b^jy^i) + \frac{b^2}{\alpha^2}y^iy^j \quad (17)$$

Thus along F^{n-1} (c), (17) and (16) lead to

gijbibj = b2.

Therefore we get i.e. Again from (17) and (18) we get

$$b_i(x(u)) = \sqrt{b^2}N_i,\; b^2 = a^{ij}b_ib_j.$$

$$b_i(x(u)) = \sqrt{b^2}N_i, where\ b\ is\ the\ length\ of\ vector\ b^i \quad (18)$$

√__

bi = aijb =b2Ni + b2α−1yi. (19) *j*

Thus we have

Theorem 4.1 In the special Finsler hypersurface F^{n-1} (c), the induced Rie-mannian metric is given by (15) and the scalar function b(x) is given by (18) and (19).

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