On The Hypersurface of a Finsler Space with Some Special Cases

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Abstract

The study of special Finsler spaces has been introduced by M. Matsumoto [4]. The purpose of the present paper is to study hypersurfaces of special Finsler spaces and also to investigate the various kinds of hypersurfaces of Finsler space with special (α,β) metric.

Keywords: Hypersurfaces, finsler spaces, C-reducible, quasi-C-reducible, preducible.

1. INTRODUCTION

The study of spaces endowed with generalized metrics was initiated by P. Finsler in 1918. The theory of hypersurfaces in general depends to a large extent on the study of the behavior of curves in them. The authors G.M. Brown, Moor, C. Shibata, M. Matsumoto, B.Y. Chen, C.S. Bagewadi, L.M. Abatangelo, Dragomir and S. Hojo have studied different properties of subspaces of Finsler, Kahler and Riemannian spaces. The concept of the (α,β)-metric \( L(α,β) \) was introduced by M. Matsumoto [5] and has been studied by many authors [1], [2], [12]. The study of some well known (α,β)-metrics, the Randers metric \( α + β \), the Kropina metric \( α^2/β \) and the generalized Kropina metric \( α^{(m+1)/m} \) have greatly contributed to the growth of Finsler geometry and its applications to theory of relativity.

2. PRELIMINARY NOTES

Let \( F^n = (M^n,L) \) be a Finsler space on a differentiable manifold \( M \) endowed with a fundamental function \( L(x,y) \). By a Finsler space, we mean a triple \( F^n = (M,D,L) \), where \( M \) denotes n-dimensional differentiable manifold, \( D \) is an open subset of a tangent vector bundle TM endowed with the differentiable structure induced by the differentiable structure of the manifold TM and \( L(x,y) : D \rightarrow R \) is a differentiable mapping having the properties

1. \( L(x,y) > 0 \), for \((x,y) \in D\),
2. \( L(x,λy) = |λ|L(x,y) \), for any \((x,y) \in D \) and \( λ \in R \) such that \((x,λy) \in D\),
3. The d-tensor field \( g_{ij} = \frac{1}{2}(\partial_i \partial_j L^2) \), for \((x,y) \in D\), where \( \partial_i = \partial / \partial x^i \), is non degenerate on \( D \).

We have the following identities (see [1] and [11]):

\[ g_{ij} = \frac{1}{2}∂_i ∂_j L^2, \quad g^{ij} = (g_{ij})^{(-1)}, \quad ∂_i = ∂ / (∂ y^i) \]

\[ C_{ij} = \frac{1}{2}∂_k g_{ij}, \quad \alpha^k = \frac{1}{2}g^k m(∂_m g_{ij}), \quad h_{ij} = g_{ij} - l_il_j \]

\[ γ^i_{jkl} = \frac{1}{2}g^{ir} (∂_j g_{rk} + ∂_k g_{jr} - ∂_r g_{jk}) \]

\[ G^i_j = \frac{1}{2}γ^i_{jk} y^j k, \quad G_i^j = ∂_k G_i^j \]

\[ G_i^j_{jk} = ∂_k G_{ij} (j), \quad G_i^{jk} = ∂_k G_i^j_{jk} \]

\[ F_{ij}^k = \frac{1}{2}g^{ir} (δ_j g_{rk} - δ_i g_{jr} - δ_r g_{jk}) \]

\[ P_{ijk} = u(δ_j C_{ijk} + C_{ijr} C_k r), \quad S_{ijk} = u(δ_j C_{ijk} + C_{ijr} C_k r) \]

Where \( δ_j = ∂ - G^r_j ∂_r \), the index 0 means contraction by \( y^i \) and the notation \( u(δ_j) \) denotes the interchange of indices \( j,k \) and subtraction. Also, we devoted to a special Finsler space \( F^n = (M^n,L) \) with a metric

\[ L(αβ) = α + β \]

\[ L_0 = \frac{α^{(n+1)} - β^{(n+1)}}{α^n}, \quad I_β^α = \frac{(α^n + (n+1) β^n)}{α^n} \]
\[ L_{\alpha\beta} = \frac{n(n + 1)}{\alpha^{n+1}}, \quad L_{\beta\gamma} = \frac{n(n + 1)}{\alpha^n} \]  
\[ L_{\alpha} = \frac{-n(n + 1)}{\alpha^{n+1}}, \]

Where

\[ L_{\alpha} = \frac{\partial L}{\partial \alpha}, \quad L_{\beta\gamma} = \frac{\partial L_{\alpha}}{\partial \beta}, \quad L_{\alpha\alpha} = \frac{\partial L_{\alpha}}{\partial \alpha} \]

In the special Finsler space \( F^n = (M^p, L) \) the normalized element of support \( l_i = \partial_i L \) and the angular metric tensor \( h_{ij} \) are given by [10]:

\[ l_i = \alpha^{-1} \partial_i L_Y + \partial_i L, \]

The fundamental tensor and its reciprocal tensor \( g_{ij} \) is given by [10]

\[ g_{ij} = \partial_i L Y_j + \partial_i L Y_j, \]

\[ \delta = \frac{p(n + 2)}{\alpha^{n+1}}, \]

3. HYPERSURFACE OF THE SPECIAL FINSLER SPACES

Now we consider the special Finsler spaces like P-reducible, quasi-C-reducible, and C-reducible. Then we prove all these special Finsler space are well-defined in Finsler hypersurface \( F^{n-1} \) under some conditions.

Definition 3.1 (see [7]) A Finsler space \( F^n \) is called a P-reducible, if the torsion tensor \( P_{ijk} \) is written as

\[ P_{ijk} = \frac{1}{\alpha} \partial_i L_{jk} \]

The fundamental tensor \( g_{ij} = \frac{1}{\alpha} \partial_i L_{jk} \) and its reciprocal tensor \( g^i_j \) is given by [10]

\[ g_{ij} = \partial_i L_{jk} + \partial_j L_{ik} + \partial_k L_{ij}, \]

where we set

\[ P_{i} = P_{im} = C'_{i/0} \]

Contracting (8) by \( B_{ij} \) and using \( h_{\alpha\beta} = g_{\alpha\beta} - l_{\alpha} l_{\beta} \), and \( h_{\alpha\beta} = h_{\alpha\beta} \) we obtain

\[ P_{ijk} B_{ijk} = (h_{ijk} + h_{ij} P_k + h_{ik} P_j)/(n + 1), \]

where we set \( P_{i} B_{ij} = P_{\alpha} = C_{\alpha/0} \). Hence we have the following result.

Theorem 3.2. A hypersurface of a P-reducible Finsler space is P-reducible.

Next we consider the curvature tensor of \( F^n \)
\[ Sh_{ijk} = ChkrCijr - ChjrCikr. \]

Contracting above equation by \( B_{ab} B_{ijk} \) and using (1), we have
\[
S_{hijk} B_{ab} B_{ijk} = (C_{hkr} C_{ijr} - C_{hj} r C_{ikr}) B_{ab} \gamma = Chkr Cijr B_{oab} \gamma - Chjr Cikr B_{oab} \gamma.
\]

This leads to:
\[ S_{aβγ} = C_{ηθαβγ} - C_{ηθαβρ}. \]

Hence \( S_{aβγ} \) is the curvature tensor of \( F_{αβ} \).

**Definition 3.3** (see [9]) A Finsler space \( F_0 (n > 2) \) is called a quasi-C-reducible, if the torsion tensor \( C_{ijk} \) is written as
\[
C_{ijk} = A_{ij} C_k + A_{jk} C_i + A_{ki} C_j,
\]
(10)

where \( A_{ij} \) is a symmetric Finsler tensor field satisfying \( A_{ij} = A_{ji} = 0 \). Contracting (9) by projection factor \( B_{ijk} \), we obtain
\[
C_{ijk B_{ab} B_{ijk}} = A_{ij} B_{ab} B_{ijk} - A_{jk} B_{ab} B_{ijk} - A_{ki} B_{ab} B_{ijk}.
\]

By using the notations on Finsler hypersurface (see[3],[8],[11]): We obtain
\[
Caβγ = AαβCγ + AαγCβ,
\]
where we setting \( A_{ab} = A_{a} B_{ab} \) is a symmetric Finsler tensor field on hypersurface \( F_{αβ} \). Thus we have:

**Theorem 3.4** A hypersurface \( F_{αβ} \) of a quasi-C-reducible Finsler space \( F_0 \) is quasi-C-reducible.

Suppose we assume that \( C_{αβγ} = 0 \), that implies
\[
C_{ijk B_{ab} B_{ijk}} = 0
\]
(11)

it means that \( C_{i} \) is tangential to the hypersurface \( F_{αβ} \).

Therefore we get
\[
S_{αβγ} = C_{ηθαβγ} = 0, \text{ hence we have:
}

**Theorem 3.7** (see [13]) A hypersurface of a C-reducible Finsler space is C-reducible.

Using the condition (10) in (12), we state that the following result:

**Theorem 3.8** A hypersurface \( F_{αβ} \) of a C-reducible Finsler space is Riemannian, if the torsion vector \( C_{i} \) is tangential to hypersurface \( F_{αβ} \).

4. **HYPERSURFACE \( F_{αβ} \) OF THE SPECIAL FINSLER SPACE**

Let us consider special Finsler metric \( L = α x^β + \frac{β n+1}{α^n} \)
with a gradient \( b(x) = c \) for a scalar function \( b(x) \) and a hypersurface \( F_{αβ} \) given by the equation \( b(x) = c \) (constant) [14].

From parametric equation \( x' = x'(u') \) of \( F_{αβ} \), we get
\[
b_1 B_{α} B_{α} = 0 \text{ and } b_2 B_{α} B_{β} = 0
\]
(14)

The induced metric \( L(u,v) \) of \( F_{αβ} \) is given by
\[
L(u,v) = a_β = a_α B_{α} B_{β}
\]
(15)

which is a Riemannian metric.

At a point of \( F_{αβ} \), from (4),(6) and (8), we have
\[
p = 1, q_0 = 0, q_1 = 0, q_2 = -α^{-2}, p_0 = 1, p_1 = α^{-1}, p_2 = 0, \zeta = 1, S_1 = 0, S_2 = -b_β / α.
\]
(16)

Therefore from (7) we get
\[
g^{ij} = α^i j - \frac{1}{α}(b_1 y^1 + b_2 y^2) + \frac{b_2^2}{α^2} y^1 y^2
\]
(17)

Thus along \( F_{αβ} \), (17) and (16) lead to
\[
gijb = b_2.
\]

Therefore we get i.e. Again from (17) and (18) we get
\[
b_1 (x(u)) = √b_2 N_1, \quad b_2 = a_β b_β.
\]
\[
b_1 (x(u)) = √b_2 N_1, \quad \text{where } b \text{ is the length of vector } b
\]
(18)

\[
b_1 = ai b = b_2 N_1 + b_2 α^{-1} y_i
\]
(19)

Thus we have.
Theorem 4.1 In the special Finsler hypersurface $F^{n-1}(c)$, the induced Riemannian metric is given by (15) and the scalar function $b(x)$ is given by (18) and (19).

REFERENCES


