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Level Crossings of a Class of Random Algebraic Polynomial

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Abstract

We know the expected number of times that a polynomial of degree n with independent normally distributed random real coefficients asymptotically crosses the line mx, when m is any real value such that $(m^2/n)\to 0$ as $n\to\infty$. The present paper shows that for m>exp (nf), where f is any function of n such that $f(n)\to\infty$, this expected number of crossings reduces to only one.

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1. INTRODUCTION

Consider the algebraic polynomial

$$P(x) = \sum_{i=0}^{n=1} a_t x^i \qquad (1.1)$$

where $a_0, a_1, a_2, \ldots..a_{n-1}$ is a sequence of independent, normally distributed random variables with mathematical exception μ and variance unity. The set of equations y=P(x) represents a family of curves in the xy-plane, Kac[7], shows that for μ=0 the number of times that this family crosses the line x-axis, on a average is (2/π) log n. Later Ibragimov and Maslova [5] obtained the same average number of crossings when they considered the case of coefficients belongings to the domain of attraction of the normal law with mean μ=0. They also showed that when μ=0 the number of crossings reduces by half. Farahmand studied the number of times that this family of curves crosses the level K=0(√n) (crosses with line y=K) for μ=0 and showed that these numbers decreases as K increases. He also showed that even in this case for μthe number of crossings reduces by half. Denote by N_m (a,b)= N (a,b) the number of times that this family crosses the line y=mx where m is any constant independent of x and let EN (a,b) be its expectation. For m-0(√n) an asymptotic value for EN(-∞,∞) was obtained by Farahamand, with which the reader will be assumed to be familiar. As noted in the latter there is a sizeable number of crossings even when the line tends to be perpendicular to the x axis, that is for m (=0 (√n)→∞ as n→∞. In this work we study the case when m is very large compared with n, and show that the number of crossings of this family of curves with such a line reduces to one. We prove.

THEOREM **-** If the coefficients of P(x) in (1,1) are independent normally distributed random variables with mean zero and variance unity, then for any constant m such that $|m| > \exp(\mathrm{nf})$, where f is any function of n such that f(n) tends to infinity as n tends to infinity, the mathematical expectation of the number of real roots of the equation P(x) = mx is asymptotic to one.

2. PROOF OF THE THEOREM

First we find a lower estimates for EN (-∞,∞). Let m>exp (n,f), then since for $|x| < 1$ the polynomial P(x) is convergent, with probability one, for x=1/2, say and n sufficiently large.

$$P(x) = mx < P(1/2) - (1/2)\exp(nf) < 0$$
$$\textit{and}\ \text{also for x} = -1/2$$
$$P(x) = -mx > P(-1/2) + (1/2)\exp(nf) > 0$$

Therefore, by the intermediate value theorem, there exists at least one real root for the function P(x)-mx in (-1/2, ½). Similarly, if m<-exp (nf) we can show that the function P(x)-mx takes on the opposite sign at x=1/2 and x=-1/2, therefore, there exists at least one real root. Hence EN (-

$\infty,\infty) \geq 1$ and we only have to show that the upper limit is one as well. We also note that both aj and –aj (i=0,1,2,….n-1) have standard normal distribution hence changing x to –x leaves the distribution of the coefficients invariant, thus EN (-∞,0)= EN (-∞,0). So we only have to consider the interval (0,∞). In by using the expected number of level crossings (page 285), the Kac-Rice formula for the equation P(x)-mx=0 is found.

$$EN(a,b) = \int_a^b [\Delta^{1/2}/\pi\alpha)\exp\{(-\alpha m^2 + 2m^2\gamma x - \beta m^2 x^2)/2\Delta$$
$$+ (\sqrt{2/n})m(\gamma x - \alpha)|\alpha^{-a/2}\exp(m_2 x^2/2\alpha)erf$$
$$|m(x-\alpha)(\alpha\Delta)^{-1/2}\} dx$$
$$= \int_a^b I_1(x)dx + \int_a^b I_2(x)dx \qquad \text{say} \qquad (2.1)$$

Where

$$\alpha = \sum_{t=\theta}^{n=1} x^{2i}, \beta = \sum_{t=\theta}^{n=1} i^2 x^{2i-2}$$
$$\gamma = \sum_{t=1}^{n=1} i x^{2i-1}, \Delta = \alpha\beta = \gamma^2 \qquad (2.2)$$

and

$$erf(x) = \int_0^x \exp(-y^2)\,dy$$

First we show $\int_0^1 I_1(x)$ dx tends to zero as n→∞. Let a be constant independent of x in the interval (0,1). For $0 \leq x \leq 1 - n^{-\alpha}$ and sufficiently large n we have

$$\gamma = \{(n-1)x^{2n+1} - nx^{2n-1} + x\}(1-x^2)^{-2}$$
$$= x(1-x^{2n})(1-x^2)^{-2} + 0\{n^{1+\alpha}\exp(-2n^{1-\alpha})\} \ldots\ldots\ldots\ldots\ldots\ldots(2.3)$$

and

$$\mathrm{erf} = \int_0^x \exp(-y^2)dy$$

First we show that $\int_0^1 I_1(x)dx$ tends to zero as n→∞. Let a be a constant independent of x in the interval (0,1). For $0 \leq x \leq 1-n^{-a}$ and all sufficiently large n we have

$$\gamma = \{(n-1)x^{2n+1} - nx^{2n-1} + x\}(1-x^2)^{-2}$$
$$= x(1-x^{2n})(1-x^2)^{-2} + 0\{n^{1+\alpha}\exp(-2n^{1-\alpha})\} \ldots\ldots\ldots\ldots\ldots\ldots(2.3)$$

And

$$\beta = (1+x^2)(1-x^{2n})(1-x2)^{-3} + 0\{n^{2+a}\exp(-2n^{1-a}) \qquad (2.4)$$

From (2.3) and (2.4) we can obtain

$$\Delta = (1-x^{2n})(1-x^2)^{-4} + 0\{n^{2+2n}\exp(-2n^{1-a}) \quad (2.5)$$

Now we choose a=1-{log log $(n)^{10}$} /log n. Then since, for all sufficiently large n,

$$n^{2+a}\exp(-2n^{1-a}) = n^{2+2n}\exp(-2\log(n)^{10}\} = n^{-18+a} \to 0.$$

All the error terms that appear in the formulas (2.3) to (2.5) will tend to zero. Hence from (2.1), (2.3), (2.4), (2.5) and since for all x

$$(1-x^2)^{2/2} - x^2(1-x^2) + x^2(1+x^2)/2 > 1/5$$

we have

$$\int_0^{1-n^{-a}} I_1(x)dx = \int_0^{1-n^{-a}} (\Delta^{1/2}/w\alpha)\exp[-m^2\{(1-x^2)/(1-x^{2n})\}$$
$$\text{x}\ \{-\text{x}^2\}/2 - x2(1-x^2)\}$$
$$+ \text{x}^2(1+\text{x}^2)/2\{(1+0\{\text{n}^{2+n}\exp(-2\text{n}^{1-a})\}\text{dx}$$

$$\leq (1/\pi)\int_0^{1-n^{-a}} (1-x^2)^{-1}\exp\{-\text{m}^2(1-\text{x}^2)/5\}\text{dx}$$
$$\leq (1/\pi)\exp(-m^2/2n^a)\int_0^{1-n^{-a}} (1-x^2)^{-1}\text{dx}$$
$$\leq (1/2\pi)\exp\{-(\text{m}^2/2\text{n})\log(\text{n})^{10}\}\log\{\text{n}^{-a}(1-\text{n}^{-a})\}$$
$$\leq (a/2\pi)\log\text{n}\exp\{-(\text{m}^2/2\text{n})\log(\text{n})^{10}\}.$$
$$\leq (a/2\pi)\exp\{\log\log\text{n}\exp-(\text{m}^2/2\text{n})\log(\text{n})^{10}\}$$

Now we note that since m>exp (n/logn) the term m^2/n tends to infinity much faster than log log n as n→∞, hence from (2.6) we can obtain

$$\int_0^{1-n^{-a}} I_1(x)dx \to 0 \text{ as } n \to \infty. \qquad (2.7)$$

To show that $\int_0^{1-n^{-a}} I_1(x)dx \to 0 \text{ as } n \to \infty.$ we first prove that $(\alpha - 2\gamma x + \beta x^2)/\Delta$ is positive for $1-n^{-a} \leq x \leq 1.$ For all sufficiently large n from (2.2) we have

$$\alpha - 2\gamma x + \beta x^2 \geq \beta x^2 - 2\gamma x$$
$$\geq n^2 x^{2n}(1-x^2)2 - 2nx^{2n+2}(1-x^2)\}$$
$$+ x^2(1+x^2)(1-x^{2n})\}(1-x^2) - 3 - n(n+1)$$
$$\geq n^3\{\log(n)^{10}\}^{-3} - 2n^2 > n^2 \qquad (2.8)$$

since $(1-x^2)^3<(2n^{-a})^3$ and $2nx^{2n+2}(1-x^2)\to$ as n→∞. Hence from (2.8) and since $\Delta<n^4$ we have

$$(\alpha - 2\gamma x + \beta x^2)/\Delta > n^{-2} \quad (2.9)$$

So from (2.9) and since from (page 319) $(\Delta^2/a) < (2n-1)^{1/2}(1-x)^{-1/2}$, we have

$$\int_0^{1-n^{-a}} I_1(x)dx < (2n-1)^{1/2}\exp(-m^2/n^2)\int_0^{1-n^{-a}}(1-x)^{-1/2}dx$$

$$< 3n^{(1-a)}/2\exp(-m^2/n^2) \quad (2.10)$$

Which tends to zero as n→∞

In order to find

$\int_0^{1-n^{-a}} I_1(x)dx$ we let y = 1/x, and divide the interval $0 \le y \le 1$ into three subintervals (0,b), (b,1-1/nd) and (1-1/nd,1) where d={(3/8) log log (n)$^{1/3}$ and b=(m^{-2} n d log n)$^{1/(4n-8)}$,. We show that in these three subintervals

$$\int_0^{1-n^{-a}} I_1(x)dx = \int_0^1 y^{-2} I_1(1/y)\,dy = \log\{(1+b)/(1-b)\} \to 0 \text{ as } n \to \infty.$$

$$(2.11)$$

For b $\le y \le 1-1/nd$ *from* (2.2) we have

$$\alpha - 2\gamma/\gamma + \beta/\gamma^2 = 1 + \sum_{i=2}^{n} y^{-2i} + \sum_{i=2}^{n} y^{-2i}(i^2 - 2i) > n^3/4 \quad (2.12)$$

and

$$\Delta = \{1-h^2(y)\}(1-y^{2n})2/y^{4n-8}(1-y^2)4 \le n^2/b^{4n-8}(1-y^2)^2$$
$$< 3n^4d^2/b^{4n-8} \quad (2.13)$$

Where

$$h(y) = ny^{n-1}(1-y^2)/(1-y^{2n})$$

Hence from (2.12) and (2.13) we can write

$$\int_0^{1-1nd} y^{-2}I_1(1/y)dy \le \exp(-m^2b^{4n-8}/12nd^2)\int_0^{1-1nd}(1-y^{-2})dy$$
$$\le (1/2)\log\ (2nd)\ \exp(-\log n/12\ d) \quad (2.14)$$

which tends to zero as n tends to infinity. Finally, as for (2.10), we obtain

$$\left[\int_{1-1/nd}^{1} y^{-2}I_1(1/y)dy < (2n-1)^{1/2}\int_{1-1/nd}^{1} y^{-2}I_1(1/y)dy < (2n-1)^{1/2}dy \atop < 2(2/d)^{1/2} \to 0 \text{ as } n \to \infty.\right] \quad 2.15)$$

Hence from (2.7), (2.10), (2.11), (2.14) and (2.15) we have

$$\int_0^\infty I_1(x)dx \to 0 \text{ as } n \to \infty. \quad (2.16)$$

In what follows we will find an upper estimate for $\int_0^\infty I_2(x)dx$. From (2.2) we have

$$\left|m(\gamma x - \alpha)\alpha^{-3/2}\right| = \left|m\{(2x^2-1)(1-x^{2n})\right|$$
$$-nx^{2n}(1-x^2)\}/(1-x^2)^{1/2}(1-x^{2n})^{3/2} \quad (2.17)$$

Now for

$$0 \le x \le \sqrt{3/2} \text{ for all } n \ge 22$$

nx^{2n} (1-x2)<n(3/4)^n<1/n

and

$$\left|(2x^2-1)(1-x^{2n})\right| \le \left|(2x^2-1)\right| < 1/n$$

only for (1-1/n)/2 <x^2<(1+1/n)/2. Let

$$\xi = \{(1-1/n)/2\}^{1/2} \text{ and } \xi' = \{(1+1/n)/2\}^{1/2}$$

then from (2.17) and for all sufficiently large n we have

$$\int_{\xi}^{\xi'} I2(x) < (4|m|/n\sqrt{n})\exp(-m^2/2n)$$

which tends to zero as n tends to infinity. On the other hand, let (x) dx indicate the integral over $\alpha < (1-1/n)/2 \le x \le (1+1/n)/2 < b$ *excluding* $(1-1/n)/2 \le x \le (1+1/n)/2.$

Let u=mxα$^{-1/2}$, since da/dx=2y and erf (x) $< \sqrt{\pi/2}$ from (2.17) we have

$$\int_0^1 I_2(x) \le (2n)^{-1/2}\int_0^{m/\sqrt{n}}\exp(-u^2/2)du < 1/2 \quad (2.18)$$

To prove that

$\int_0^\infty I_1(x)dx \to 0$ *as* n → ∞, *from* (2.2) for y = 1/x we have.

$$(\gamma x - \alpha)/\alpha^{1/2}\Delta^{1/2} = \gamma^{2n-1}(1-y^2)^{3/2}\{-1-ny-y^{2n-1}(1-y^2)^2\}$$
$$-y^{2n-4}\}\{1-h^2(y)\}^{-1/2}(1-y^{2n})^{-1/2} \quad (2.19)$$

And

$$x^2/\alpha = y^{2n-4}(1-y^2)/(1-y^{2n}) \quad (2.20)$$

First we let

$0 \le y \le (mn^2)^{-1/(2n-1)}$ then y → 0 as n → ∞, from (2.19) we have

$$(x - \alpha)\alpha^{1/2}\Delta^{1/2} < 2ny^{2n-1} \quad 2.21)$$

Let u

$m x\alpha^{-1/2}$ *and* $\lambda = (mn^2)^{-1/(2n-3)}$, then from (2.1),(2.20) and (2.21) we obtain

$$\int_0^{\lambda} y^{-2} I_2 (1/y)\, dy < \operatorname{erf}(1/n) \int_0^{\infty} \exp(-u^2/2) \to 0 \text{ as } n \to \infty. \qquad (2.22)$$

Since erf (1/n)→0 for all sufficiently large n. On the other hand for

$\lambda \le y \le 1$ *we* let $u = mx\alpha^{-1/2}$, sin *ce*

$$y^{2n-4}(1-y^2)/(1-y^{2n}) > 1/(2mn^2)$$

from (2.20) we have

$$\int_0^1 y^{-2} I_2 (1/y)\, dy < \int_{m/2n2}^{m/\sqrt{n}} \exp(-u^2/2)du$$

$$\le (m/\sqrt{n} - m^2/2n^2)\exp(-m^2/4n^4) \to 0 \text{ as } n \to \infty. \qquad (2.23)$$

Hence from (2.22) and (2.23) we have

$$\int_1^{\infty} I_2 (x)\, dx \to 0 \text{ as } n \to \infty. \qquad (2.24)$$

Finally from (2.7), (2.10), (2.16), (2.18) and (2.24) we obtain

$EN(0,\infty) \le 1/2$

and since $EN(-\infty,\infty) = 2EN(0,\infty)$

Remark and open Problem: The asymptotic number of crossings of the polynomial P(x) with line mx decreases as m=0(√n) increases. In this paper we proved that when $|m \ge \exp(nf)|$ the number of crossings reduces to one. The behaviour of the number of crossings between these two lines is not known. A subsequent study could consider the case when (m^2/n) tends to any non zero constant as n tends to infinity and as a guessed target $EN(-\infty,\infty) \sim (1/\pi)\log n$, which is half the number of crossings when m=0 seems reasonable. (Knowing a rough value for $EN(-\infty,\infty)$ is useful in order to sufficient upper and lower bounds for $EN(-\infty,\infty)$ leading to an asymptotic formula). Indeed, the behaviour of $EN(-\infty,\infty)$ for other values of m is also interesting, but it will involve more analysis especially for the $\int_{-\infty}^{\infty} I_2(x)\, dx$ part of $EN(-\infty,\infty)$.

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