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Nilpotent and Solvable Lie Algebroids

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Abstract

In this paper, we present new example of solvable Lie algebra which is not nilpotent Lie algebra. Also we define the nilpotent an solvable Lie algebroid and we show that every Lie algebroid E is nilpotent (solvable) Lie algebroid if and only if the dual of it be a nilpotent (solvable) Lie algebroid.

Keywords: *Nilpotent and Solvable Lie algebra, Lie algebroid and its dual, Connection.*

1. INTRODUCTION

The theory of Lie algebras first appeared in the work of Norwegian mathematician, Sophus Lie. It appeared infinitesimal transformations in his attempt to simplify difficult problems he encountered in continuous groups. This theory was later named after Sophos Lie by Herman Weyl in the 1930s. Then the study of Lie algebras has remained and active area of research. Before the development of the theory of Liealgebras, Galois and Abel had already studied the solvable and nilpotent groups, [11].Lie algebroids were first introduced and studied by J.Pradines, following work by C.Ehresmann and P.Liberman on differentiable groupoids (Later called Lie groupoids). Just as Lie algebras infinitesimal objects of Lie groups, Lie algebroids are the infinitesimal objects of Lie groupoids. [12]

In the first section of this paper, we discuss the structure of the nilpotent and solvable Lie algebra, we show that there exists a solvable Lie algebra such that is not nilpotent, but we know that every nilpotent Lie algebra is solvable Lie algebra. Section two is devoted to definitions of nilpotent and solvable Lie algebroid. By the duality of Lie algebroid we prove two crucial theorems which ones show that every Lie algebroid is nilpotent (solvable) Lie algebroid if and only if dual of it is nilpotent (solvable) Lie algebroid, another theorem characterize the dual of Lie algebroid, by connection of its. Also in this section some new examples of this category are presented.

Now we call the derived series of ideals.

Definition 1.1 For Liealgebra g, define the series of ideals $D^i g$ (called derived series) by:

$$D^0 g = g \text{ and } D^{i+1} g = [D^i g, D^i g]$$

Also we have the following central series:

Definition 1.2 For Liealgebra g, define the series of ideals $D_i g \subset g$ (called lower central series) by:

$$D_0 g = g \text{ and } D_{i+1} g = [g, D_i g]$$

In mathematics, a Lie algebra is nilpotent if the lower central series becomes zero eventually, i.e.

Definition 1.3 ([7]) The Lie algebra g is called nilpotent if there is number k such that for all $n \geq k$, $D_n g = 0$.

Example 1.4 Let

$$L = \left\{ A_{n\times n} \middle| \begin{array}{c} A_{n\times n} \text{ be triangle matrices with} \\ \text{zero elements in original diameter} \end{array} \right\}$$

And

$$[L, L] \coloneqq A \times B - B \times A.$$

Then L is nilpotent Lie algebra.

Also, Lie algebra g is solvable if its derived series terminates in the zero subalgebra, i.e.

Definition 1.5 ([7,4,13]) Lie algebra g is called solvable if there is a number k such that for all $n \geq k$, $D^n g = 0$.

Lemma 1.6 Every nilpotent Lie algebra is solvable.

Note that it is easy to show that every nilpotent Lie algebra is solvable, but the converse of this subject is not correct, for example:

Example 1.7 The up triangle matrices with non-zero elements on original diameter are solvable but is not nilpotent Lie algebra.

2. Nilpotent and Solvable Lie Algebroid

Definition 2.1 ([6]) Let M be n-dimensional manifold and its tangent bundle be (TM,π_M,M). A Lie algebroid over the manifold M is the triple $(E,[.,.],\sigma)$, where $\pi: E\to M$ isvector bundle of rank m over M, whose $C^\infty M$-module of sections, $\sec(E)$ is equipped with a Lie algebra structure $[.,.]$ and $\sigma: E\to TM$ is a vector bundle homomorphism (called anchor) which induces a Lie algebra homomorphism (also denoted σ) from $\sec(E)$ to $X(M)$, satisfying compatibility conditions, for every $f\in C^\infty M$ and $s_1,s_2\in \sec(E)$then:

$$[s_1,fs_2]=f[s_1,s_2]+(\sigma(s_1)f)s_2.$$

We take the local coordinates (x^i) on an open $U\subset M$, a local basis $\{s_\alpha\}$ of sections of the of the bundle $\pi^{-1}U\to U$ generates local coordinates (x^i,y^α)on E. The local functions $\sigma_\alpha^i(x), L_{\alpha\beta}^\gamma(x)$ on M given by:

$$\sigma(s_\alpha)=\sigma_\alpha^i\frac{\partial}{\partial x^i},$$

$$[s_\alpha,s_\beta]=L_{\alpha\beta}^\gamma s_\gamma,$$

where $i=1,\dots,n$; $\alpha,\beta=1,\dots,m$.

Now we define a nilpotent (solvable) Lie algebroid:

Definition 2.2 Lie algebroid E is nilpotent (solvable) if $sec(E)$ be a nilpotent (solvable) Lie algebra.

Example 2.3 The triple $(TM,[.,.],M)$ where $\pi: TM\to M$ is tangent bundle, anchor $\sigma: TM\to TM$ is identity and $[.,.]$ is Lie bracket of vector fields. TM is a nilpotent Lie algebroid basis on M. Above Lie algebroid is called trivial Lie algebroid.

Example 2.4 Every nilpotent (solvable) Lie algebra$g$is nilpotent (solvable) Lie algebroid.

Lemma 2.5 Every nilpotent Lie algebroid is solvable.

Proof: By the lemma 1.6 and definition of Lie algebroid, every nilpotent Lie algebroid is solvable Lie algebroid.

Example 2.6 In example 2.3 $(TM,[.,.],M)$ is a solvable Lie algebroid.

Definition 2.7 ([5,8,9]) Let $\tau: E^*\to M$ be the dual of$\pi: E\to M$and $(E,[.,.],\sigma)$ be a Lie algebroid structure over M. One can construct a Lie algebroid structure over E^*, by taking the prolongation of $(E,[.,.],\sigma)$over E^*that calls dual of Lie algebroid E. This structure is given by the following objects:

1) The associated vector bundle is $(\mathbf{T}E^*,\tau_1,E^*)$, where

$$\mathbf{T}E^*=\cup_{u^*\in E^*}\mathbf{T}_{u^*}E^*$$

And

$$\mathbf{T}_{u^*}E^*=\{(u_x,v_{u^*})\in E_x\times T_{u^*}E^*|\sigma(u_x)=T_{\tau(u^*)}\tau(v_{u^*}),\tau(u^*)=x\in M\}$$
,

also the projection $\tau_1:\mathbf{T}E^*\to E^*$ defined by $\tau_1(u_x,v_{u^*})=u^*$.

2) The Lie algebra structure $[.,.]$ on $\sec(\mathbf{T}E^*)$ is defined by the following way:

If $\rho_1,\rho_2\in\sec(\mathbf{T}E^*)$, such that

$$\rho_i(u^*)=\Big(X_i\big(\tau(u^*)\big),U_i(u^*)\Big), \text{ where}$$

$$X_i\in\sec(E), U_i\in X(E^*)\text{and}\sigma\Big(X_i\big(\tau(u^*)\big)\Big)=T_{u^*}\tau\big(U_i(u^*)\big)$$

$$for\ \ i=1,2.$$

$$\text{Then}[\rho_1,\rho_2](u^*)=\Big([X_1,X_2]\big(\tau(u^*)\big),[U_1,U_2](u^*)\Big).$$

3) The anchor is the projection $\sigma^1:\mathbf{T}E^*\to TE^*$, $\sigma^1(u,v)=v$.

Note: If$\mathrm{T}_\tau:\mathbf{T}E^*\to E, \mathrm{T}_\tau(u,v)=u$ then $(V\mathbf{T}E^*,\tau_1|_{V\mathbf{T}E^*},E^*)$ with $V\mathbf{T}E^*:=Ker\ \mathrm{T}_\tau$ is subbundle of $(\mathbf{T}E^*,\tau_1,E^*)$, called the vertical subbundle. If (x^i,μ_α) are local coordinates on E^* at u^* and $\{s_\alpha\}$ is local basis of sections of $\pi: E\to M$, then the local basis of $sec(\mathbf{T}E^*)$ is $\{\chi_\alpha,\rho^\alpha\}$, where

$$\chi_\alpha(u^*)=\left(s_\alpha\big(\tau(u^*)\big),\sigma_\alpha^i\frac{\partial}{\partial x^i}|_{u^*}\right),\rho^\alpha(u^*)=\left(0,\frac{\partial}{\partial\mu_\alpha}|_{u^*}\right).$$

The Lie bracket on the elements of this basis is:

$$[\chi_\alpha,\chi_\beta]=L_{\alpha\beta}^\gamma\chi_\gamma\ ,[\chi_\alpha,\rho^\alpha]=0\ ,[\rho^\beta,\rho^\alpha]=0.$$

One important theorem in this section is:

Theorem 2.8 Lie algebroid E is nilpotent (solvable) if and only if the dual of E is nilpotent (solvable). Its may the nilpotent degree (solvable degree) E and its dual are not equal.

Proof: Let E be a nilpotent Lie algebroid, then by the definition there exist k such that for all $\geq k$,$D_n(\sec(E))=0$. For $k=4$ we show that if $D_4(\sec(E))=0$, then there exist$l$ such that $D_l(\sec(\mathbf{T}E^*))=0$.

Let $\rho_1,\rho_2,\rho_3,\rho_4,\rho_5\in\sec(\mathbf{T}E^*)$,$\rho_1=(X_1,U_1),\rho_2=(X_2,U_2),\rho_3=(X_3,U_3),\rho_4=(X_4,U_4),\rho_5=(X_5,U_5)$and

$D_4(\sec(\mathbf{T}E^*)) = \left[\sec(\mathbf{T}E^*),\left[\sec(\mathbf{T}E^*),\left[\sec(\mathbf{T}E^*),[\sec(\mathbf{T}E^*),\sec(\mathbf{T}E^*)]\right]\right]\right].$

Then for all $\rho_i, i = 1, \dots, 5$ we have

$$D_4(\sec(\mathbf{T}E^*)) = \left[\rho_1,\left[\rho_2,\left[\rho_3,[\rho_4,\rho_5]\right]\right]\right]$$
$$= \left(\left[X_1,\left[X_2,\left[X_3,[X_4,X_5]\right]\right]\right],\left[U_1,\left[U_2,\left[U_3,[U_4,U_5]\right]\right]\right]\right).$$

The Lie bracket $\left[X_1,\left[X_2,\left[X_3,[X_4,X_5]\right]\right]\right]$ is zero (because E is nilpotent with degree 4) and U_i are vector field, so there are nilpotent Lie algebra, therefore the dual of E is nilpotent Lie algebroid. The converse of this subject is similar to. For other k the similar processing is satisfied.

Example 2.9 In example 2.3 we show that TM is nilpotent Lie algebroid, in this example we show that dual of TM is nilpotent Lie algebroid. We know that $\tau: T^*M \to M$ (cotangent bundle of M) is dual of tangent bundle $\pi: TM \to M$. At first we define dual of Lie algebroid $TM$by $\mathbf{T}T^*M$.

$$\mathbf{T}T^*M = \cup_{\omega \in T^*M} \mathbf{T}_\omega T^*M$$

Where

$\mathbf{T}_\omega T^*M = \{(X_p, v_\omega) \in T_pM \times T_\omega(T^*M) | T_p(\tau(v_\omega)) = X_p\}$, and anchor map $\sigma^1: \mathbf{T}T^*M \to T^*M$ defined by$(X_p, v_\omega) \to \omega$.

Let $\rho_i \in \sec(\mathbf{T}T^*M), \rho_i = (X_i, U_i)$, such that $X_i \in \sec(TM), U_i \in X(T^*M)$ so X_i, U_i are vector fields, since every vector field is nilpotent Lie algebra, the X_i andU_i are nilpotent Lie algebras, then$\mathbf{T}T^*M$ is nilpotent Lie algebroid.

Example 2.10 We know that every Lie algebra g is a Lie algebroid basis on set $\{x\}$ such that $x \in g$ and anchor is zero map. Let $g$be nilpotent Lie algebra, we show that dual of g is nilpotent Lie algebroid.

At first we define the dual of:

Let $\pi: g \to \{x\}$ be a vector bundle and zero map σ be an anchor, the dual of π is $\tau: g^* \to \{x\}$ such that the fibers are dual of fibers of π. We define

$\mathbf{T}g^* = \cup_{u^* \in g^*} \mathbf{T}_{u^*} g^*$,

Where

$\mathbf{T}_{u^*} g^* = \{(u_x, v_{u^*}) \in g_x \times T_{u^*}g^* | \sigma(u_x) = T_{\tau(u^*)}\tau(v_{u^*}), \tau(u^*) = x\}$,

we have $g_x = g$, so

$\mathbf{T}_{u^*} g^* = \{(g, v_{u^*}) \in g \times T_{u^*}g^* | T_g\tau(v_{u^*}) = 0\}$.

Therefore $\cup_{u^* \in g^*} \mathbf{T}_{u^*} g^* = \cup \{(g, v_{u^*}) \in g \times T_{u^*}g^*\} = g \times \cup_{u^* \in g^*} T_{u^*}g^* = g \times Tg^*$ then $\mathbf{T}g^* = g \times Tg^*$ and anchor is

$$\sigma^1: \mathbf{T}g^* \to g^*, (g, v_{u^*}) \to u^*.$$

Let $\rho_1, \rho_2 \in \sec(\boldsymbol{T}g^*), \rho_1 = (X_1, U_1), \rho_2 = (X_2, U_2)$ such that $X_i$be a section of g, $X_i: \{x\} \to g$ and $U_i \in X(g^*)$. We suppose that $g$is nilpotent Lie algebra, so there is k such that for all $n \geq k, D_n g = 0$, let $k = 4$, then $\left[g,\left[g,\left[g,[g,g]\right]\right]\right] = 0$ by the definition $X_i(x) \in g$

we have: $\left[X_1(x),\left[X_2(x),\left[X_3(x),[X_4(x),X_5(x)]\right]\right]\right] = 0$ for all $X_i(x), i = 1, \dots, 5$, U_i are vector fields, so there are nilpotent Lie algebras, thus there is a l such that $D_l(\sec(\boldsymbol{T}g^*)) = 0$ and $\boldsymbol{T}g^*$ is nilpotent Lie algebroid.

Definition 2.11 ([6]) A connection on $\boldsymbol{T}E^*$ is an almost product structure $\mathbb{N}$on$\tau_1: \boldsymbol{T}E^* \to E^*$(i.e. a bundle morphism $\mathbb{N}: \boldsymbol{T}E^* \to \boldsymbol{T}E^*$, such that $\mathbb{N}^2 = id$) smooth on $\boldsymbol{T}E^* \backslash \{0\}$ such that $V\boldsymbol{T}E^* = \ker(id + \mathbb{N})$.

1) If $\mathbb{N}$is a connection on E^* , then $H\boldsymbol{T}E^* = \ker(id - \mathbb{N})$ is horizontal subbundle associated to $\mathbb{N}$and $\boldsymbol{T}E^* = V\boldsymbol{T}E^* \oplus H\boldsymbol{T}E^*$.

2) A connection $\mathbb{N}$on $\boldsymbol{T}E^*$induce two projection$h, v: \boldsymbol{T}E^* \to \boldsymbol{T}E^*$ such that $h(\rho) = \rho^h, v(\rho) = \rho^v$, for every $\rho \in \sec(\boldsymbol{T}E^*)$. We have
$$h = \frac{1}{2}(id + \mathbb{N}), v = \frac{1}{2}(id - \mathbb{N}),$$
$\ker h = \mathrm{Im} v = V\boldsymbol{T}E^*, Im\, h = \ker v = H\boldsymbol{T}E^*$.

3) A locally connection can be expressed as $\mathbb{N}(\chi_\alpha) = \chi_\alpha + 2\mathbb{N}_{\alpha\beta}\rho^\alpha, \mathbb{N}(\rho^\alpha) = -\rho^\alpha$, where $\mathbb{N}_{\alpha\beta}$ are local coefficients on α, β. The vector fields $\delta^*_\alpha = h(\chi_\alpha) = \chi_\alpha + \mathbb{N}_{\alpha\beta}\rho^\beta$ generates a basis of $H\boldsymbol{T}E^*$ and ρ^α generates a basis of $V\boldsymbol{T}E^*$.

The second important theorem is:

Theorems 2.12$TE^*$is nilpotent (solvable) Lie algebroid if and only if $H\boldsymbol{T}E^*$is nilpotent (solvable) sub Lie algebroid.

Proof: We can write dual of Liealgebroid$E$, $\boldsymbol{T}E^*$ as

$$\boldsymbol{T}E^* = V\boldsymbol{T}E^* \oplus H\boldsymbol{T}E^*$$

Where $V\boldsymbol{T}E^*$is vertical sub bundle and $H\boldsymbol{T}E^*$ is horizontal sub bundle of $\boldsymbol{T}E^*$. We know that $\{\chi_\alpha, \rho^\alpha\}$is basis of $\sec(\boldsymbol{T}E^*)$ such that ρ^α is a basis for $V\boldsymbol{T}E^*$, also $\delta^*_\alpha = \chi_\alpha + \mathbb{N}_{\alpha\beta}\rho^\beta$, ($\mathbb{N}_{\alpha\beta}$ *is coefficient*) is basis for $H\boldsymbol{T}E^*$. By the definition of Lie bracket on $\sec(\boldsymbol{T}E^*)$, $V\boldsymbol{T}E^*$is nilpotent sub Lie algebroid. Therefore $\boldsymbol{T}E^*$is nilpotent Lie algebroid if and only if $H\boldsymbol{T}E^*$is nilpotent sub Lie algebroid.

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