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On The Convolution of Gamma Random Variables With Respect To Two Parameters Majorization

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Abstract

Let X_{α_i,λ_i}, $i=1,...,n$ are independent random variables from a distribution function gamma(α_i,λ_i), $i=1,...,n$. We survey the likelihood ratio ordering of the convolution of the gamma random variables in terms of the majorization order of the two parameters holds. For $\alpha_i,\alpha_i^*\geq 1$, if $(\frac{1}{\alpha_1^*},...,\frac{1}{\alpha_n^*})\overset{m}{\succeq}(\frac{1}{\alpha_1},...,\frac{1}{\alpha_n})$ and $(\lambda_1^*,...,\lambda_n^*)\overset{m}{\succeq}(\lambda_1,...,\lambda_n)$, we show that $\sum_{i=1}^{n} X_{\alpha_i^*,\lambda_i^*}$ is larger than $\sum_{i=1}^{n} X_{\alpha_i,\lambda_i}$ according to the likelihood ratio ordering. We find the useful bounds for the survival function and hazard rate function of convolution of non-identical gamma random variables.

Keywords: *Convolution, likelihood ratio ordering, two-parameter majorization, gamma distribution, totally positivity.*

1. INTRODUCTION

There are many applications of Convolution of independent random variables in reliability, optics, acoustics, electrical engineering, physics, quality control and insurance. (See Alzaid and Kayid (2009), Nadarajah and Dey (2005), Mukherjee (2007), Killmann and Collani (2001)).

Stochastic ordering, hazard rate ordering, and likelihood ratio ordering have been proven to be very useful in applied probability, statistics, reliability, survival analysis. Some relevant references are Boland et al. (1994), Korwar (2002), Khaledi and Kochar (2002, 2004, and 2006), Fathi Manesh and Khaledi (2008), Zhao and Balakrishnan (2009 and 2010). Gamma distribution is one of the most popular distributions in statistics, engineering and reliability. In this paper, we will concentrate on likelihood ratio ordering of convolution of independent gamma random variables in terms of majorization order of scale and inverse of shape parameter holds.

Let $X_1,...,X_n$ be a random sample from a gamma distribution with shape parameter α, and scale parameter λ and with density function

$$f_{\alpha,\lambda}(x)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{(\alpha-1)}e^{-\lambda x},\quad \alpha>0,\lambda>0,x>0. \qquad (1.1)$$

Let $X_{(\alpha_i,\lambda_i)}$ and $X_{(\alpha_i^*,\lambda_i^*)}$ have gamma distribution with parameters (α_i,λ_i) and (α_i^*,λ_i^*), $i=1,...n$ respectively and if $(\frac{1}{\alpha_1^*},...,\frac{1}{\alpha_n^*})$ majorize $(\frac{1}{\alpha_1},...,\frac{1}{\alpha_n})$, and $(\lambda_1^*,...,\lambda_n^*)$ majorize $(\lambda_1,...,\lambda_n)$ then we consider the relation between $\sum_{i=1}^{n}X_{(\alpha_i,\lambda_i)}$ and $\sum_{i=1}^{n}X_{(\alpha_i^*,\lambda_i^*)}$ according to the likelihood ratio ordering.

First we review the necessary definitions and concepts.

Assume random variables X and Y have density functions f and g, survival functions $\overline{F}$ and $\overline{G}$, respectively and let u_x and u_y denote the upper end points of the support of X and Y, respectively.

If $\overline{F}(x)\leq\overline{G}(x)$ for all X, the random variable X is said to be stochastically smaller than Y, (denoted by $X\leq_{st}Y$). X is said to be smaller than Y in hazard rate ordering if $\overline{G}(x)/\overline{F}(x)$ is increasing in $x\in(-\infty,\max(u_x,u_y))$ (denoted by $X\leq_{hr}Y$). If $g(x)/f(x)$ is increasing in $x\in(-\infty,\max(u_x,u_y))$ the random variable X is said to be smaller than Y in likelihood ratio ordering (denoted by $X \le_{lr} Y$). (See Boland et al, (1994), Shaked and Shanthikumar (2007)).

Let $\{a_{(1)} \le ... \le a_{(n)}\}$ and $\{b_{(1)} \le ... \le b_{(n)}\}$ denote the increasing arrangement of the components of a vector $\underline{a} = (a_1,...,a_n)$ and $\underline{b} = (b_1,...,b_n)$. A vector $\underline{a}$ is said to be majorized by $\underline{b}$ ($\underline{b}$ majorize $\underline{a}$, $\underline{b} \overset{m}{\succeq} \underline{a}$) if $\sum_{i=1}^{k} a_{(i)} \ge \sum_{i=1}^{k} b_{(i)}$ for $k = 1,...,n\text{-}1$ and $\sum_{i=1}^{n} a_{(i)} = \sum_{i=1}^{n} b_{(i)}$ (Marshall and Olkin (2011)).

A function $f(x,\theta)$, $x \in \chi$ and $\theta \in \Theta$ where χ and Θ are subsets of the real line, is said to be totally positive of order 2, denoted by $\mathrm{TP}_2(x,\theta)$ if

$$\frac{f(x,\theta^*)}{f(x,\theta)} \le \frac{f(x^*,\theta^*)}{f(x^*,\theta)} \tag{1.1}$$

for all $x < x^*$ in χ and $\theta < \theta^*$ in Θ.

2. MAIN RESULT

To prove the main theorem in this section we shall need the following lemma.

The convolutions of the gamma distributions with respect to the likelihood ratio order for either a common shape parameter or a common scale parameter has studied by Korwar (2002)

Lemma 2.1 (Karlin (1968)). Let A, B and C be subsets of the real line and let $L(x,z)$ be TP_2 for $x \in \mathrm{A}$, $z \in \mathrm{B}$ and $M(z,y)$ be TP_2 for $z \in \mathrm{B}$ and $y \in \mathrm{C}$. Then

$$K(x,y) = \int_{\mathrm{B}} L(x,z)M(z,y)d\mu(z)$$

is TP_2 for $x \in \mathrm{A}$ and $y \in \mathrm{C}$. Here μ is sigma-finite measure on R.

Lemma 2.2 (Korwar (2002)). Let $X_{\alpha,\lambda_1},...,X_{\alpha,\lambda_n}$ be independent gamma random variables with parameters α and λ_i, $i = 1,...,n$, and let $X_{\alpha,\lambda_1^*},...,X_{\alpha,\lambda_n^*}$ be another set of independent gamma random variables with parameters α and λ_i^*, $i = 1,...,n$ independent from the first set. If $\alpha \ge 1$ then

$$(\lambda_1^*,...,\lambda_n^*) \overset{m}{\succeq} (\lambda_1,...,\lambda_n) \Rightarrow \sum_{i=1}^{n} X_{\alpha,\lambda_i^*} \ge_{lr} \sum_{i=1}^{n} X_{\alpha,\lambda_i}, \tag{2.1}$$

Lemma 2.3. Let $X_{\alpha_1,\lambda},...,X_{\alpha_n,\lambda}$ be independent random variables from Gamma distributions with parameters λ and α_i ($\alpha_i > 1$), $i = 1,...,n$ and $X_{\alpha_1^*,\lambda},...,X_{\alpha_n^*,\lambda}$ be another set of independent gamma random variables with parameters α_i and λ_i, $i = 1,...,n$, independent from the first set. Then

$$\left(\frac{1}{\alpha_1^*},...,\frac{1}{\alpha_n^*}\right) \overset{m}{\succeq} \left(\frac{1}{\alpha_1},...,\frac{1}{\alpha_n}\right) \Rightarrow \sum_{i=1}^{n} X_{\alpha_i^*,\lambda} \ge_{lr} \sum_{i=1}^{n} X_{\alpha_i,\lambda}. \tag{2.2}$$

Proof. Let $X_{\alpha_i,\lambda} \sim \mathrm{Gamma}(\alpha_i,\lambda)$, $i = 1,...,n$ and $h(\omega,\sum_{i=1}^{n}\alpha_i,\lambda)$ be distribution of $\sum_{i=1}^{n} X_{\alpha_i,\lambda}$ and let $X_{\alpha_i^*,\lambda} \sim \mathrm{Gamma}(\alpha_i^*,\lambda)$, $i = 1,...,n$ be another set of independent random variables, then $\sum_{i=1}^{n} X_{\alpha_i,\lambda}$ $(\sum_{i=1}^{n} X_{\alpha_i^*,\lambda})$ has a gamma distribution with parameters $\sum_{i=1}^{n}\alpha_i$ $(\sum_{i=1}^{n}\alpha_i^*)$ and λ, if $(\frac{1}{\alpha_1^*},...,\frac{1}{\alpha_n^*}) \overset{m}{\succeq} (\frac{1}{\alpha_1},...,\frac{1}{\alpha_n})$ for $\alpha_i,\alpha_i^* \ge 1$ then $\alpha_1^* + ... + \alpha_n^* \ge \alpha_1 + ... + \alpha_n$, therefore

$$\frac{h(\omega,\sum_{i=1}^{n}\alpha_i^*,\lambda)}{h(\omega,\sum_{i=1}^{n}\alpha_i,\lambda)} = \frac{\Gamma(\sum_{i=1}^{n}\alpha_i)}{\Gamma(\sum_{i=1}^{n}\alpha_i^*)}\omega^{\sum_{i=1}^{n}\alpha_i^* - \sum_{i=1}^{n}\alpha_i}$$

is increasing in ω and hence $\sum_{i=1}^{n} X_{\alpha_i^*,\lambda} \ge_{lr} \sum_{i=1}^{n} X_{\alpha_i,\lambda}$

Theorem 2.1. Let $X_{\alpha_1,\lambda_1}, X_{\alpha_2,\lambda_2}$ be independent random variables from Gamma distributions with parameters α_i and λ_i, $i = 1,2$ and let $X_{\alpha_1,\lambda_1}, X_{\alpha_2,\lambda_2}$ be another set of independent random variables independent from the first set. If $\alpha_i,\alpha_i^* \ge 1$ then,

$$\left.\begin{array}{l}(\frac{1}{\alpha_1^*},\frac{1}{\alpha_2^*}) \overset{m}{\succeq} (\frac{1}{\alpha_1},\frac{1}{\alpha_2}) \\ (\lambda_1^*,\lambda_2^*) \overset{m}{\succeq} (\lambda_1,\lambda_2)\end{array}\right\} \Rightarrow X_{\alpha_1^*,\lambda_1^*} + X_{\alpha_2^*,\lambda_2^*} \ge_{lr} X_{\alpha_1,\lambda_1} + X_{\alpha_2,\lambda_2}, \tag{2.3}$$

Proof. Let $X_{\alpha,\lambda_i} \sim \mathrm{Gamma}(\alpha,\lambda_i)$, $i = 1,2$ be independent gamma random variables and let $X_{\alpha,\lambda_i^*} \sim \mathrm{Gamma}(\alpha,\lambda_i^*)$, $i = 1,2$ be another set of independent gamma random variables independent from the first set. If $\alpha \ge 1$ then by Lemma 2.2

$$(\lambda_1^*,\lambda_2^*) \overset{m}{\succeq} (\lambda_1,\lambda_2) \Rightarrow X_{\alpha,\lambda_1^*} + X_{\alpha,\lambda_2^*} \ge_{lr} X_{\alpha,\lambda_1} + X_{\alpha,\lambda_2},$$

Let for $\alpha \ge 1$, $\lambda_1 + \lambda_2 = \lambda_1^* + \lambda_2^* = c$, without loss of generality assume that $\lambda_2 < \lambda_1$ and $\lambda_2^* < \lambda_1^*$, from which it follows that $\lambda_2^* \le \lambda_2 \le \lambda_1 \le \lambda_1^*$ and $\lambda_1,\lambda_1^* \in \left[\frac{c}{2},2\right)$, and let

$h(\omega;\alpha,\lambda)$ be distribution of $X_{\alpha,\lambda_1}+X_{\alpha,\lambda_2}$ then $\frac{h(\omega;\alpha,\lambda^*)}{h(\omega;\alpha,\lambda)}$ is increasing in ω or for $\lambda^*\geq\lambda$ and $\omega^*\geq\omega$

$$\frac{h(\omega;\alpha,\lambda^*)}{h(\omega;\alpha,\lambda)}\leq\frac{h(\omega^*;\alpha,\lambda^*)}{h(\omega^*;\alpha,\lambda)}$$

then

$$\frac{h(\omega^*;\alpha,\lambda)}{h(\omega;\alpha,\lambda)}\leq\frac{h(\omega^*;\alpha,\lambda^*)}{h(\omega;\alpha,\lambda^*)} \tag{2.4}$$

Again let $X_{\alpha_i,\lambda}\sim\text{Gamma}(\alpha_i,\lambda)$, $i=1,2$ be independent gamma random variables and let $X_{\alpha_i^*,\lambda}\sim\text{Gamma}(\alpha_i^*,\lambda)$, $i=1,2$ be another set of independent gamma random variables independent from the first set. If $\alpha_i,\alpha_i^*\geq 1$ then by Lemma 2.3

$$(\frac{1}{\alpha_1^*},\frac{1}{\alpha_2^*})\overset{m}{\succeq}(\frac{1}{\alpha_1},\frac{1}{\alpha_2})\Rightarrow X_{\alpha_1^*,\lambda}+X_{\alpha_2^*,\lambda}\geq_{lr}X_{\alpha_1,\lambda}+X_{\alpha_2,\lambda}$$

Let $(\frac{1}{\alpha_1^*},\frac{1}{\alpha_2^*})\overset{m}{\succeq}(\frac{1}{\alpha_1},\frac{1}{\alpha_2})$ for $\alpha_i,\alpha_i^*\geq 1$, $i=1,2$ then $\alpha^*=\alpha_1^*+\alpha_2^*\geq\alpha_1+\alpha_2=\alpha$, if $h(\omega;\alpha_1+\alpha_2,\lambda)$ denote the distribution function of $X_{\alpha_1,\lambda}+X_{\alpha_2,\lambda}$, then for $\omega^*\geq\omega$

$$\frac{h(\omega;\alpha^*,\lambda)}{h(\omega;\alpha,\lambda)}\leq\frac{h(\omega^*;\alpha^*,\lambda)}{h(\omega^*;\alpha,\lambda)}$$

or

$$\frac{h(\omega^*;\alpha,\lambda)}{h(\omega;\alpha,\lambda)}\leq\frac{h(\omega^*;\alpha^*,\lambda)}{h(\omega;\alpha^*,\lambda)} \tag{2.5}$$

(2.4) is correct for any fixed α (α^*), then from (2.4) and (2.5)

$$\frac{h(\omega^*;\alpha,\lambda)}{h(\omega;\alpha,\lambda)}\leq\frac{h(\omega^*;\alpha^*,\lambda)}{h(\omega;\alpha^*,\lambda)}\leq\frac{h(\omega^*;\alpha^*,\lambda^*)}{h(\omega;\alpha^*,\lambda^*)},$$

and hence

$$\frac{h(\omega;\alpha^*,\lambda^*)}{h(\omega;\alpha,\lambda)}\leq\frac{h(\omega^*;\alpha^*,\lambda^*)}{h(\omega^*;\alpha,\lambda)},$$

or

$$X_{\alpha_1^*,\lambda_1^*}+X_{\alpha_2^*,\lambda_2^*}\geq_{lr}X_{\alpha_1,\lambda_1}+X_{\alpha_2,\lambda_2}$$

In next theorem we extend Theorem 2.1 for a sum of n independent gamma random variables.

Theorem 2.2. Let $X_{\alpha_1,\lambda_1},\ldots,X_{\alpha_n,\lambda_n}$ be independent random variables from gamma distribution functions with parameters α_i and λ_i, $i=1,\ldots,n$, and let $X_{\alpha_1,\lambda_1},\ldots,X_{\alpha_n,\lambda_n}$ be another set of independent gamma random variables independent from the first set. If $\alpha_i,\alpha_i^*\geq 1$ then

$$\left.\begin{array}{l}(\frac{1}{\alpha_1^*},\ldots,\frac{1}{\alpha_n^*})\overset{m}{\succeq}(\frac{1}{\alpha_1},\ldots,\frac{1}{\alpha_n})\\(\lambda_1^*,\ldots,\lambda_n^*)\overset{m}{\succeq}(\lambda_1,\ldots,\lambda_n)\end{array}\right\}\Rightarrow\sum_{i=1}^{n}X_{\alpha_i^*,\lambda_i^*}\geq_{lr}\sum_{i=1}^{n}X_{\alpha_i,\lambda_i}, \tag{2.6}$$

Proof. For $\alpha_i\geq 1$ let $(\frac{1}{\alpha_1^*},\frac{1}{\alpha_2^*})\overset{m}{\succeq}(\frac{1}{\alpha_1},\frac{1}{\alpha_2})$ and $\alpha_i=\alpha_i^*$, $i=3,\ldots,n$ and $(\lambda_1^*,\lambda_2^*)\overset{m}{\succeq}(\lambda_1,\lambda_2)$ and $\lambda_i=\lambda_i^*$, $i=3,\ldots,n$. It follows from (2.3)

$$X_{\alpha_1^*,\lambda_1^*}+X_{\alpha_2^*,\lambda_2^*}\geq_{lr}X_{\alpha_1,\lambda_1}+X_{\alpha_2,\lambda_2}.$$

If $\alpha_i\geq 1$, X_{α_i,λ_i} has a log-concave density then the random variable $S((\alpha_3^*,\lambda_3^*),\ldots,(\alpha_n^*,\lambda_n^*))=\sum_{i=3}^{n}X_{\alpha_i^*,\lambda_i^*}$ has log-concave density, since the convolution of r.v.s. with log-concave densities has a log-concave density (cf. Dharmadhikari and Joag-dev, 1988, p.17); and $S((\alpha_3^*,\lambda_3^*),\ldots,(\alpha_n^*,\lambda_n^*))$ is independent of X_{α_1,λ_1}, X_{α_2,λ_2}, $X_{\alpha_1^*,\lambda_1^*}$ and $X_{\alpha_2^*,\lambda_2^*}$. Using the Lemma 2.1 we obtain that

$$S((\alpha_1^*,\lambda_1^*),(\alpha_2^*,\lambda_2^*),(\alpha_3^*,\lambda_3^*)\ldots,(\alpha_n^*,\lambda_n^*))\geq_{lr}S((\alpha_1,\lambda_1),(\alpha_2,\lambda_2),(\alpha_3^*,\lambda_3^*)\ldots,(\alpha_n^*,\lambda_n^*))$$

By using Lemma 2B.1 of Marshal and Olkin (1979) proof is complete.

Remark 2.1. Let $X_{\alpha_1,\lambda_1},\ldots,X_{\alpha_n,\lambda_n}$ be independent random variables from Gamma distributions with parameters α_i and λ_i, and let $\bar{\alpha}=\frac{1}{n}\sum_{i=1}^{n}\alpha_i$, $\overline{(1/\alpha)}=\frac{1}{n}\sum_{i=1}^{n}1/\alpha_i$ and $\bar{\lambda}=\frac{1}{n}\sum_{i=1}^{n}\lambda_i$, then $(\frac{1}{\alpha_1},\ldots,\frac{1}{\alpha_n})\overset{m}{\succeq}(\overline{(1/\alpha)},\ldots,\overline{(1/\alpha)})$, $(\alpha_1,\ldots,\alpha_1)\overset{m}{\succeq}(\bar{\alpha},\ldots,\bar{\alpha})$, and $(\lambda_1,\ldots,\lambda_n)\overset{m}{\succeq}(\bar{\lambda},\ldots,\bar{\lambda})$ then from (2.6) for $\alpha=1/\overline{(1/\alpha)}$,

$$\sum_{i=1}^{n}X_{\alpha_i,\lambda_i}\geq_{lr}\sum_{i=1}^{n}X_{\alpha,\bar{\lambda}}. \tag{2.7}$$

Similarly from Zhao (2011),

$$\sum_{i=1}^{n}X_{\alpha_i,\lambda_i}\geq_{lr}\sum_{i=1}^{n}X_{\bar{\alpha},\bar{\lambda}}. \tag{2.8}$$

Now $\sum_{i=1}^{n}X_{\bar{\alpha},\bar{\lambda}}\sim\Gamma(n\bar{\alpha},\bar{\lambda})$ and $\sum_{i=1}^{n}X_{\alpha,\bar{\lambda}}\sim\Gamma(n\alpha,\bar{\lambda})$, so for $\alpha_i\geq 1$, $n\bar{\alpha}\geq n\alpha$ and hence $\sum_{i=1}^{n}X_{\bar{\alpha},\bar{\lambda}}\geq_{lr}\sum_{i=1}^{n}X_{\alpha,\bar{\lambda}}$.

Shaked and Shanthikumar (2007) have shown that the following implications among the stochastic orderings

$$X\leq_{lr}Y\Rightarrow X\leq_{hr}Y\Rightarrow X\leq_{st}Y.$$

Now by using (2.7) and (2.8) we can find the useful bounds for the survival function and hazard rate function of convolution of non-identical gamma random variables with different shape and scale parameters. See Figure 1 and Figure 2.

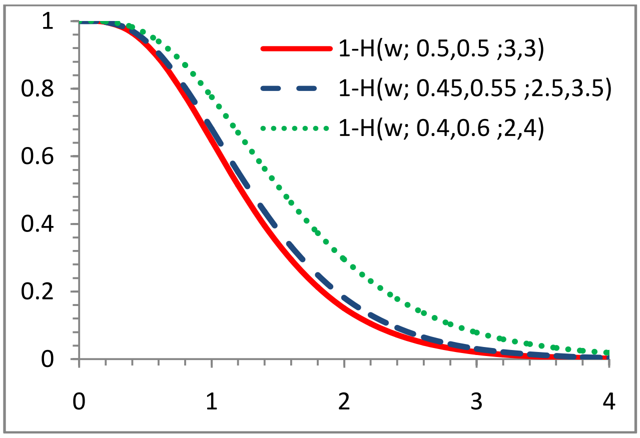

Figure 1: Survival function of $S_{\omega}(\frac{1}{\alpha_1},\frac{1}{\alpha_2};\lambda_1,\lambda_2)$.

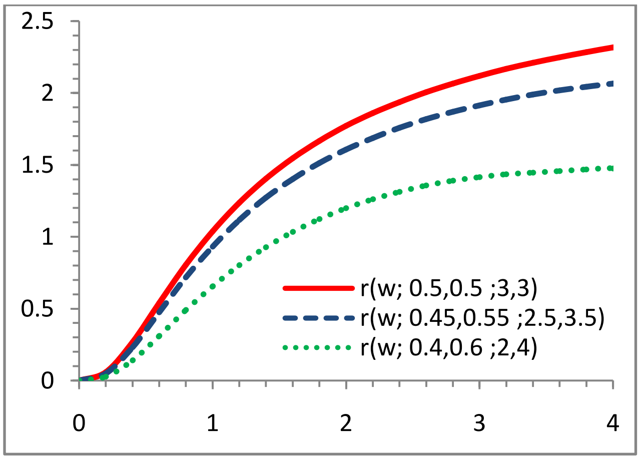

Figure 2: Hazard rate function of $S_{\omega}(\frac{1}{\alpha_1},\frac{1}{\alpha_2};\lambda_1,\lambda_2)$

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