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# On The Convolution of Gamma Random Variables With Respect To Two Parameters Majorization

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***Abstract***

*Let $X_{\alpha_i,\lambda_i}, i=1,...,n$ are independent random variables from a distribution function $gamma(\alpha_i,\lambda_i)$, $i=1,...,n$. We survey the likelihood ratio ordering of the convolution of the gamma random variables in terms of the majorization order of the two parameters holds. For $\alpha_i,\alpha_i^* \geq 1$, if $(\frac{1}{\alpha_1^*},...,\frac{1}{\alpha_n^*}) \succeq^m (\frac{1}{\alpha_1},...,\frac{1}{\alpha_n})$ and $(\frac{1}{\alpha_1^*},...,\frac{1}{\alpha_n^*}) \succeq^m (\frac{1}{\alpha_1},...,\frac{1}{\alpha_n})$, we show that $\sum_{i=1}^n X_{\alpha_i^*,\lambda_i^*}$ is larger than $\sum_{i=1}^n X_{\alpha_i,\lambda_i}$ according to the likelihood ratio ordering.*

***Keywords:*** *Convolution, likelihood ratio ordering, two-parameter majorization, gamma distribution, totally positivity.*

## 1.INTRODUCTION

There are many applications of Convolution of independent random variables in reliability, optics, acoustics, electrical engineering, physics, quality control and insurance and the gamma distribution is one of the most popular distributions in statistics, engineering and reliability that convolution of gamma random variable have many applications in reliability, optics, acoustics, electrical engineering, physics and insurance mathematics, insurance problems, quality control has been studied by several authors. (See Alzaid and Kayid (2009), Nadarajah and Dey (2005), Mukherjee (2007), Killmann and Collani (2001)).

Stochastic ordering, hazard rate ordering, and likelihood ratio ordering have been proven to be very useful in applied probability, statistics, reliability, survival analysis.

Some relevant references are Boland et al. (1994), Korwar (2002), Khaledi and Kochar(2002,2004, and 2006), FathiManesh and Khaledi (2008),Zhao and Balakrishnan(2009 and 2010).Sometimes we need to consider simultaneous majorization of the more than one parameter. In this paper, we will concentrate on likelihood ratio ordering of convolution of independent gamma random variables in terms of majorization order ofscale and inverse of shape parameter holds.

Let $X_1,...,X_n$ be a random sample from a gamma distribution with shape parameter $\alpha$, and scale parameter $\lambda$ and with density function

$$f_{\alpha,\lambda}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{(\alpha-1)} e^{-\lambda x}, \quad \alpha > 0, \lambda > 0, x > 0. \qquad (1.1)$$

Let $X_{(\alpha_i,\lambda_i)}$ and $X_{(\alpha_i^*,\lambda_i^*)}$ have gamma distribution with parameters $(\alpha_i,\lambda_i)$ and $(\alpha_i^*,\lambda_i^*)$ respectively, $i=1,...n$, and if $(\frac{1}{\alpha_1^*},...,\frac{1}{\alpha_n^*})$ majorizes (or is majorized by) $(\frac{1}{\alpha_1},...,\frac{1}{\alpha_n})$ and $(\lambda_1^*,...,\lambda_n^*)$ majorizes (or is majorized by) $(\lambda_1,...,\lambda_n)$ then we consider the relation between $\sum_{i=1}^n X_{(\alpha_i,\lambda_i)}$ and $\sum_{i=1}^n X_{(\alpha_i^*,\lambda_i^*)}$ according to the likelihood ratio ordering.

First we review the necessary definitions and concepts.

Assume random variables $X$ and $Y$ have density functions $f$ and $g$, survival functions $\overline{F}$ and $\overline{G}$, respectively and let $u_x$ and $u_y$ denote the upper end points of the support of $X$ and $Y$, respectively.

If $\overline{F}(x) \le \overline{G}(x)$ for all $x$, the random variable $X$ is said to be stochastically smaller than $Y$, (denoted by $X \le_{st} Y$). $X$ is said to be smaller than $Y$ in hazard rate ordering if $\overline{G}(x)/\overline{F}(x)$ is increasing in $x \in (-\infty, \max(u_x, u_y))$ (denoted by $X \le_{hr} Y$). If $g(x)/f(x)$ is increasing in $x \in (-\infty, \max(u_x, u_y))$ the random variable $X$ is said to be smaller than $Y$ in likelihood ratio ordering (denoted by $X \le_{lr} Y$). (See Boland et al, (1994), Shaked and Shanthikumar (2007))

Let $\{a_{(1)} \le \dots \le a_{(n)}\}$ and $\{b_{(1)} \le \dots \le b_{(n)}\}$ denote the increasing arrangement of the components of a vector $\underline{a} = (a_1, \dots, a_n)$ and $\underline{b} = (b_1, \dots, b_n)$. A vector $\underline{a}$ is said to be majorized by $\underline{b}$ ($\underline{b}$ majorizes $\underline{a}$, $\underline{b} \overset{m}{\succeq} \underline{a}$) if $\sum_{i=1}^{k} a_{(i)} \ge \sum_{i=1}^{k} b_{(i)}$ for $k = 1, \dots, n-1$ and $\sum_{i=1}^{n} a_{(i)} = \sum_{i=1}^{n} b_{(i)}$ (Marshall and Olkin (2011)).

A function $f(x, \theta)$, $x \in \chi$ and $\theta \in \Theta$ where $\chi$ and $\Theta$ are subsets of the real line, is said to be totally positive of order 2, denoted by $\mathrm{TP}_2(x, \theta)$ if

$$\frac{f(x, \theta^*)}{f(x, \theta)} \le \frac{f(x^*, \theta^*)}{f(x^*, \theta)} \qquad (1.1)$$

for all $x < x^*$ in $\chi$ and $\theta < \theta^*$ in $\Theta$.

## 2. MAIN RESULT

To prove the main theorem in this section we shall need the following lemma.

The convolutions of the gamma distributions with respect to the likelihood ratio order for either a common shape parameter or a common scale parameter has studied by Korwar (2002)

**Lemma 2.1** (Karlin (1968)). Let A, B and C be subsets of the real line and let $L(x, z)$ be $\mathrm{TP}_2$ for $x \in \mathrm{A}$, $z \in \mathrm{B}$ and $M(z, y)$ be $\mathrm{TP}_2$ for $z \in \mathrm{B}$ and $y \in \mathrm{C}$. Then

$$K(x, y) = \int_{\mathrm{B}} L(x, z) M(z, y) d\mu(z)$$

is $\mathrm{TP}_2$ for $x \in \mathrm{A}$ and $y \in \mathrm{C}$. Here $\mu$ is sigma-finite measure on $\mathrm{R}$.

**Lemma 2.2** (Korwar (2002)). Let $X_{\alpha,\lambda_1}, \dots, X_{\alpha,\lambda_n}$ be independent gamma random variables with parameters $\alpha$ and $\lambda_i$, $i = 1, \dots, n$, and let $X_{\alpha,\lambda_1^*}, \dots, X_{\alpha,\lambda_n^*}$ be another set of independent gamma random variables with parameters $\alpha$ and $\lambda_i^*$, $i = 1, \dots, n$, independent from the first set. If $\alpha \ge 1$ then

$$(\lambda_1^*, \dots, \lambda_n^*) \overset{m}{\succeq} (\lambda_1, \dots, \lambda_n) \Rightarrow \sum_{i=1}^{n} X_{\alpha,\lambda_i^*} \ge_{lr} \sum_{i=1}^{n} X_{\alpha,\lambda_n}, \qquad (2.1)$$

**Lemma 2.3.** Let $X_{\alpha_1,\lambda}, \dots, X_{\alpha_n,\lambda}$ be independent random variables from Gamma distributions with parameters $\lambda$ and $\alpha_i$ $(\alpha_i > 1), i = 1, \dots, n$ and $X_{\alpha_1^*,\lambda}, \dots, X_{\alpha_n^*,\lambda}$ be another set of independent gamma random variables with parameters $\alpha_i$ and $\lambda_i$, $i = 1, \dots, n$, independent from the first set. Then

$$\left(\frac{1}{\alpha_1^*}, \dots, \frac{1}{\alpha_n^*}\right) \overset{m}{\succeq} \left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n}\right) \Rightarrow \sum_{i=1}^{n} X_{\alpha_i^*,\lambda} \ge_{lr} \sum_{i=1}^{n} X_{\alpha_i,\lambda}. \qquad (2.2)$$

**Proof.** Let $X_{\alpha_i,\lambda} \sim \mathrm{Gamma}(\alpha_i, \lambda)$, $i = 1, \dots, n$ and $h(\omega, \sum_{i=1}^{n} \alpha_i, \lambda)$ be distribution of $\sum_{i=1}^{n} X_{\alpha_i,\lambda}$ and let $X_{\alpha_i^*,\lambda} \sim \mathrm{Gamma}(\alpha_i^*, \lambda)$, $i = 1, \dots, n$ be another set ofindependent random variables, then $\sum_{i=1}^{n} X_{\alpha_i,\lambda}$ ($\sum_{i=1}^{n} X_{\alpha_i^*,\lambda}$) has a gamma distribution with parameters $\sum_{i=1}^{n} \alpha_i$ ($\sum_{i=1}^{n} \alpha_i^*$) and $\lambda$, if $\left(\frac{1}{\alpha_1^*}, \dots, \frac{1}{\alpha_n^*}\right) \overset{m}{\succeq} \left(\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_n}\right)$ for $\alpha_i, \alpha_i^* > 1$ then $\alpha_1^* + \dots + \alpha_n^* \ge \alpha_1 + \dots + \alpha_n$, therefore

$$\frac{h(\omega, \sum_{i=1}^{n} \alpha_i^*, \lambda)}{h(\omega, \sum_{i=1}^{n} \alpha_i, \lambda)} = \frac{\Gamma(\sum_{i=1}^{n} \alpha_i)}{\Gamma(\sum_{i=1}^{n} \alpha_i^*)} \omega^{\sum_{i=1}^{n} \alpha_i^* - \sum_{i=1}^{n} \alpha_i}$$

is increasing in $\omega$ and hence $\sum_{i=1}^{n} X_{\alpha_i^*,\lambda} \ge_{lr} \sum_{i=1}^{n} X_{\alpha_i,\lambda}$

**Theorem 2.1.** Let $X_{\alpha_1,\lambda_1}, X_{\alpha_2,\lambda_2}$ be independent random variables from Gamma distributions with parameters $\alpha_i$ and $\lambda_i$, $i = 1, 2$ and let $X_{\alpha_1,\lambda_1}, X_{\alpha_2,\lambda_2}$ be another set of independent random variables independent from the first set. If $\alpha_i, \alpha_i^* \ge 1$ then,

$$\left.\begin{aligned} &\left(\frac{1}{\alpha_1^*}, \frac{1}{\alpha_2^*}\right) \overset{m}{\succeq} \left(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\right) \\ &(\lambda_1^*, \lambda_2^*) \overset{m}{\succeq} (\lambda_1, \lambda_2) \end{aligned}\right\} \Rightarrow X_{\alpha_1,\lambda_1^*} + X_{\alpha_2,\lambda_2^*} \ge_{lr} X_{\alpha_1^*,\lambda_1} + X_{\alpha_2^*,\lambda_2}, \qquad (2.3)$$

**Proof.** Let $X_{\alpha,\lambda_i} \sim \text{Gamma}(\alpha,\lambda_i)$, $i=1,2$ be independent gamma random variables and let $X_{\alpha,\lambda_i^*} \sim \text{Gamma}(\alpha,\lambda_i^*)$, $i=1,2$ be another set of independent gamma random variables independent from the first set. If $\alpha \geq 1$ then by Lemma 2.2

$$(\lambda_1^*,\lambda_2^*) \overset{m}{\succeq} (\lambda_1,\lambda_2) \Rightarrow X_{\alpha,\lambda_1^*} + X_{\alpha,\lambda_2^*} \geq_{lr} X_{\alpha,\lambda_1} + X_{\alpha,\lambda_2},$$

Let for $\alpha \geq 1$, $\lambda_1+\lambda_2=\lambda_1^*+\lambda_2^*=c$, without loss of generality assume that $\lambda_2<\lambda_1$ and $\lambda_2^*<\lambda_1^*$, from which and $(\lambda_1^*,\lambda_2^*) \overset{m}{\succeq} (\lambda_1,\lambda_2)$, it then follows that $\lambda_2^* \leq \lambda_2 \leq \lambda_1 \leq \lambda_1^*$, $\lambda_1^* \geq \lambda_1$ and $\lambda_1,\lambda_1^* \in \left[\frac{c}{2},2\right)$, and let $h(\omega;\alpha,\lambda)$ be distribution of $X_{\alpha,\lambda_1}+X_{\alpha,\lambda_2}$ then $\frac{h(\omega;\alpha,\lambda^*)}{h(\omega;\alpha,\lambda)}$ is increasing in $\omega$ or for $\lambda^* \geq \lambda$ and $\omega^* \geq \omega$

$$\frac{h(\omega;\alpha,\lambda^*)}{h(\omega;\alpha,\lambda)} \leq \frac{h(\omega^*;\alpha,\lambda^*)}{h(\omega^*;\alpha,\lambda)}$$

then

$$\frac{h(\omega^*;\alpha,\lambda)}{h(\omega;\alpha,\lambda)} \leq \frac{h(\omega^*;\alpha,\lambda^*)}{h(\omega;\alpha,\lambda^*)} \tag{2.4}$$

Again let $X_{\alpha_i,\lambda} \sim \text{Gamma}(\alpha_i,\lambda)$, $i=1,2$ be independent gamma random variables and let $X_{\alpha_i^*,\lambda} \sim \text{Gamma}(\alpha_i^*,\lambda)$, $i=1,2$ be another set of independent gamma random variables independent from the first set. If $\alpha_i,\alpha_i^* \geq 1$ then byLemma2.3

$$\left(\frac{1}{\alpha_1^*},\frac{1}{\alpha_2^*}\right) \overset{m}{\succeq} \left(\frac{1}{\alpha_1},\frac{1}{\alpha_2}\right) \Rightarrow X_{\alpha_1^*,\lambda}+X_{\alpha_2^*,\lambda} \geq_{lr} X_{\alpha_1,\lambda}+X_{\alpha_2,\lambda}$$

Let $\left(\frac{1}{\alpha_1^*},\frac{1}{\alpha_2^*}\right) \overset{m}{\succeq} \left(\frac{1}{\alpha_1},\frac{1}{\alpha_2}\right)$ for $\alpha_i,\alpha_i^* \geq 1$, $i=1,2$, then $\alpha^*=\alpha_1^*+\alpha_2^* \geq \alpha_1+\alpha_2=\alpha$ ,if $h(\omega;\alpha_1+\alpha_2,\lambda)$ denotethe distribution function of $X_{\alpha_1,\lambda}+X_{\alpha_2,\lambda}$, then for $\omega^* \geq \omega$

$$\frac{h(\omega;\alpha^*,\lambda)}{h(\omega;\alpha,\lambda)} \leq \frac{h(\omega^*;\alpha^*,\lambda)}{h(\omega^*;\alpha,\lambda)}$$

then

$$\frac{h(\omega^*;\alpha,\lambda)}{h(\omega;\alpha,\lambda)} \leq \frac{h(\omega^*;\alpha^*,\lambda)}{h(\omega;\alpha^*,\lambda)} \tag{2.5}$$

(2.4) is correct for any fixed $\alpha$ ($\alpha^*$), then from (2.4) and (2.5)

$$\frac{h(\omega^*;\alpha,\lambda)}{h(\omega;\alpha,\lambda)} \leq \frac{h(\omega^*;\alpha^*,\lambda)}{h(\omega;\alpha^*,\lambda)} \leq \frac{h(\omega^*;\alpha^*,\lambda^*)}{h(\omega;\alpha^*,\lambda^*)},$$

and hence

$$\frac{h(\omega;\alpha^*,\lambda^*)}{h(\omega;\alpha,\lambda)} \leq \frac{h(\omega^*;\alpha^*,\lambda^*)}{h(\omega^*;\alpha,\lambda)},$$

or

$$X_{\alpha_1^*,\lambda_1^*}+X_{\alpha_2^*,\lambda_2^*} \geq_{lr} X_{\alpha_1,\lambda_1}+X_{\alpha_2,\lambda_2}$$

In next theorem we extend Theorem 2.1 for a sum of $n$ independent gamma random variables.

**Theorem 2.2.** Let $X_{\alpha_1,\lambda_1},\ldots,X_{\alpha_n,\lambda_n}$ be independent random variables from gamma distribution functions with parameters $\alpha_i$ and $\lambda_i$, $i=1,\ldots,n$, and let $X_{\alpha_1,\lambda_1},\ldots,X_{\alpha_n,\lambda_n}$ be another set of independent gamma random variables independent from the first set. If $\alpha_i,\alpha_i^* \geq 1$ then

$$\left.\begin{array}{l}\left(\frac{1}{\alpha_1^*},\ldots,\frac{1}{\alpha_n^*}\right) \overset{m}{\succeq} \left(\frac{1}{\alpha_1},\ldots,\frac{1}{\alpha_n}\right)\\ (\lambda_1^*,\ldots,\lambda_n^*) \overset{m}{\succeq} (\lambda_1,\ldots,\lambda_n)\end{array}\right\} \Rightarrow \sum_{i=1}^{n} X_{\alpha_i^*,\lambda_i^*} \geq_{lr} \sum_{i=1}^{n} X_{\alpha_i,\lambda_i}, \tag{2.6}$$

**Proof.** For $\alpha_i \geq 1$ let $\left(\frac{1}{\alpha_1^*},\frac{1}{\alpha_2^*}\right) \overset{m}{\succeq} \left(\frac{1}{\alpha_1},\frac{1}{\alpha_2}\right)$ and $\alpha_i=\alpha_i^*$, $i=3,\ldots,n$ and $(\lambda_1^*,\lambda_2^*) \overset{m}{\succeq} (\lambda_1,\lambda_2)$ and $\lambda_i=\lambda_i^*$, $i=3,\ldots,n$. It follows from (2.5) $X_{\alpha_1^*,\lambda_1^*}+X_{\alpha_2^*,\lambda_2^*} \geq_{lr} X_{\alpha_1,\lambda_1}+X_{\alpha_2,\lambda_2}$.
If $\alpha_i \geq 1$, $X_{\alpha_i,\lambda_i}$ has a log-concave density and the random variable then the random variable $S((\alpha_3^*,\lambda_3^*),\ldots,(\alpha_n^*,\lambda_n^*)) = \sum_{i=3}^{n} X_{\alpha_i^*,\lambda_i^*}$ has log-concave density, since the convolution of r.v.s. with log-concave densities has a log-concave density (cf. Dharmadhikari and Joag-dev, 1988, p.17); and $S((\alpha_3^*,\lambda_3^*),\ldots,(\alpha_n^*,\lambda_n^*))$ is independent of $X_{\alpha_1,\lambda_1}$, $X_{\alpha_2,\lambda_2}$, $X_{\alpha_1^*,\lambda_1^*}$ and $X_{\alpha_2^*,\lambda_2^*}$. Using the Lemma 2.1 we obtain that

$$S((\alpha_1^*,\lambda_1^*),(\alpha_2^*,\lambda_2^*),(\alpha_3^*,\lambda_3^*)\ldots,(\alpha_n^*,\lambda_n^*)) \geq_{lr} S((\alpha_1,\lambda_1),(\alpha_2,\lambda_2),(\alpha_3^*,\lambda_3^*)\ldots,(\alpha_n^*,\lambda_n^*))$$

By using Lemma 2B.1 of Marshal and Olkin(1979) proof is complete.

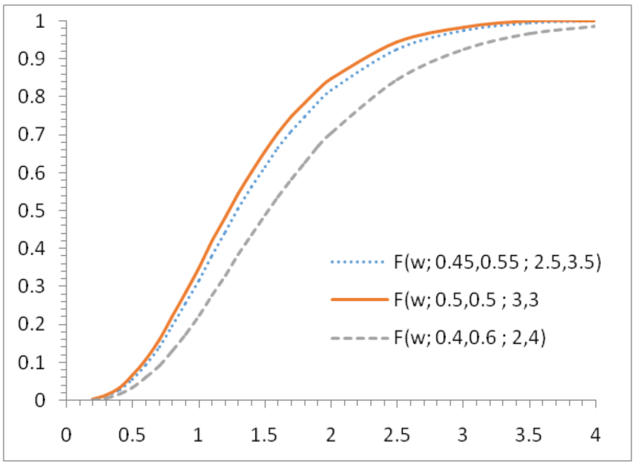


**Figure 1:**Distribution function of $S_\omega(\frac{1}{\alpha_1},\frac{1}{\alpha_2};\lambda_1,\lambda_2)$

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